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Parallel tractor extension and metrics of split G_2 holonomy

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Abstract

Parallel tractor extension
and metrics of split G_2 holonomy

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Nurowski showed that a maximally nonintegrable 2-plane field on a 5-manifold induces a natural conformal structure of signature $(2,3)$ on that manifold. We show that in the real-analytic case, applying the Fefferman-Graham ambient construction to the conformal structures produced this way on oriented manifolds always yields metrics with pseudo-Riemannian holonomy contained in the split real form G_2 of the exceptional Lie group $G_2^{\mathbb{C}}$. We furthermore show that such metrics have holonomy equal to G_2 generically, in the sense that if the holonomy of such a metric is a proper subgroup of G_2, then the 7-jet of the underlying 2-plane field at any arbitrary point must be contained in some proper subvariety of the 7-jet space there. This construction hence yields an infinite-dimensional family of metrics with holonomy equal to G_2. Both the containment and the genericity statements generalize results of Leistner and Nurowski.

To prove the containment, we prove the general result that any parallel tractor tensor on an n-dimensional, general-signature conformal structure, $n \geq 3$, admits an extension to a tensor on the ambient space parallel in a weak sense: If n is odd, there is always an extension to an tensor on the ambient space parallel to infinite order, and if n is even there is an extension parallel to order $\frac{1}{2}n - 1$. In particular, if n is odd and the underlying data is real-analytic, then there is a bona fide parallel extension for any real-analytic ambient metric. Hammerl and Sagerschnig produced for any 2-plane field on an oriented 5-manifold

a parallel tractor 3-form suitably compatible with the tractor metric, so the extension result produces a parallel 3-form on the ambient manifold compatible with the ambient metric, the existence of which is equivalent to the containment of holonomy in $\mathrm{G}_2 < \mathrm{SO}(3,4)$.

We also investigate parallel extension to infinite order when n is even, in which case existence of such extensions is generally obstructed, and we show that the parallel extension result yields a family of necessary integrability conditions for parallel tractor tensors of any type.

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DEDICATION

To Elfriede Wickerham, and to John and Catherine Willse.

0.1 Introduction

In his celebrated but difficult 1910 "five variables" paper [Car10], Cartan studied the geometry of 2-plane fields on 5-manifolds satisfying a maximal nonintegrability condition called *genericity* and solved the associated equivalence problem. This investigation included the most involved application of his equivalence method by that time, and it revealed a striking connection between these structures and the exceptional complex Lie algebra $\mathfrak{g}_2^{\mathbb{C}}$. In fact, earlier Cartan and Engel simultaneously realized $\mathfrak{g}_2^{\mathbb{C}}$ as the Lie algebra comprising the vector fields whose flows preserve a particular (complex) 2-plane field on $\mathbb{C}^5$ [Car93, Eng93]. Taking an appropriate real slice analogously yields a (real) 2-plane field on $\mathbb{R}^5$ preserved exactly by vector fields constituting a Lie algebra isomorphic to the split real form $\mathfrak{g}_2$ of $\mathfrak{g}_2^{\mathbb{C}}$. In the real setting, Cartan's solution to the above equivalence problem encodes a 2-plane field on a (real) 5-manifold M in a P-principal bundle $E \to M$ together with a Cartan connection $\omega \in \Gamma(T^*E \otimes \mathfrak{g}_2)$, and the pair (E, ω) is unique up to equivalence. Here, P is a certain group realizing a certain parabolic subalgebra $\mathfrak{p} < \mathfrak{g}_2$, and in modern language, one calls the pair (E, ω) a *parabolic geometry* of type $(\mathfrak{g}_2, P)$.

The group $\mathrm{G}_2^{\mathbb{C}}$ realizing $\mathfrak{g}_2^{\mathbb{C}}$ and its real forms G_2^c (the *compact form*) and G_2 (the *split form*) occur naturally in other fundamental settings, too. In 1955, Berger produced a list that contained all possible irreducibly acting holonomy groups of simply connected pseudo-Riemannian manifolds that are not locally symmetric [Ber55]. Later investigations shortened this list, showing that some of the groups Berger included only occurred for symmetric spaces (Theorem 1.6.4 here records this shortened list). All of $\mathrm{G}_2^{\mathbb{C}}$, G_2^c, and G_2 are on the list, and the latter two are the only groups on Berger's List that can occur in odd dimension other than the full group $\mathrm{O}(p, q)$—in both cases, only in 7 dimensions. It remained unknown whether all of the groups on the list actually occurred as the holonomy of some metric until 1987, when Bryant resolved positively the outstanding cases [Bry87, Section 5], including all three of these forms of G_2.

After this construction, metrics of holonomy G_2^c were studied intensively, including in the physics literature, because of applications to string theory and supersymmetry [Gub], and Joyce used technically demanding analytic methods to produce the first examples of compact Riemannian manifolds with that holonomy group. Metrics of holonomy equal to the split form G_2 received less attention.

Nurowski observed that, remarkably, any generic 2-plane field $\mathbb{D}$ on a 5-manifold M induces a canonical signature-$(2,3)$ conformal structure $c_{\mathbb{D}}$ on M [Nur05]: Roughly, one exploits the natural embedding $\mathrm{G}_2 \hookrightarrow \mathrm{SO}(3,4)$ to extend the Cartan connection associated to $\mathbb{D}$ equivariantly to a normal conformal connection. We call any conformal structure that arises via this construction a **Nurowski conformal structure**. Following Cartan, Bryant and Hsu gave a proof that the plane fields $\mathbb{D}$ admit a local quasi-normal form [BH93]: There always exist local coordinates (x, y, p, q, z) on M in which (the restriction of) $\mathbb{D}$ is $\operatorname{span}\{\partial_q, \partial_x + p\partial_y + q\partial_p + F\partial_z\}$; moreover, for any smooth function F on an open subset of $\mathbb{R}^5$ for which F_{qq} is nonvanishing, the corresponding 2-plane field $\mathbb{D}_F$ so defined is generic. Nurowski's construction is totally explicit in the sense that he gives a (formidable) formula (A.1) for a representative of the conformal structure induced by the plane field associated to such a function F. With Nurowski's construction in hand, one can ply the well-developed theory of conformal geometry to investigate the geometry of generic 2-plane fields on 5-manifolds.

Fefferman and Graham showed that any conformal structure c can be canonically encoded in an essentially unique *ambient metric* [FG]. First, the metric bundle $\mathcal{G} \to M$, the principal $\mathbb{R}^+$-bundle whose sections are representative metrics of the conformal structure c, enjoys a tautological, degenerate, symmetric bilinear form $\mathbf{g}$. One then constructs a signature-$(p+1, q+1)$ metric $\widetilde{g}$ on some neighborhood $\widetilde{M} \subseteq \mathcal{G} \times \mathbb{R}$ of $\mathcal{G} \times \{0\}$ (which we identify with $\mathcal{G}$ itself) that pulls back to $\mathbf{g}$ via the inclusion $\mathcal{G} \hookrightarrow \widetilde{M}$. Requiring that $\widetilde{g}$ be homogeneous and Ricci-flat is enough to guarantee weak notions of both uniqueness and existence of such a metric: If n is odd, then there is an ambient metric that is Ricci-flat to infinite order, and any two such ambient metrics agree to that order and up to a diffeomorphism that restricts to the identity on $\mathcal{G}$. The same holds for even n, except that both existence and uniqueness fail at order $n/2$.

In 2009, Leistner and Nurowski applied the ambient construction to the conformal structures $c_{\mathbb{D}}$ induced by real-analytic plane fields $\mathbb{D}$ [LN10]. (For such plane fields, one may choose the ambient metric of $c_{\mathbb{D}}$ to be real-analytic, too; such a metric is Ricci-flat and is unique up to extension and to diffeomorphisms that restrict to the identity on $\mathcal{G}$.) Specifically, they restricted attention to real-analytic 2-plane fields defined via the quasi-normal form by the polynomial functions

$$F[\mathbf{a}, b](x, y, p, q, z) = q^2 + a_0 + a_1 p + a_2 p^2 + a_3 p^3 + a_4 p^4 + a_5 p^5 + a_6 p^6 + bz \qquad (1)$$

and showed by explicitly constructing appropriate parallel objects that the (real-analytic) ambient metrics of the corresponding conformal structures have holonomy contained in G_2. They also devise specialized technical criteria (given here in Lemma 1.6.15) that a general 2-plane field $\mathbb{D}$ on a 5-manifold must satisfy if the ambient metric of the induced Nurowski conformal structure has holonomy strictly contained in G_2. Finally, they use these conditions to analyze the plane fields defined by the functions $F[\mathbf{a}, b]$ by computing explicitly tensorial data associated to representative metrics of the induced conformal structures and show that, except for $(a_0, \ldots, a_6, b)$ in an explicit, proper subvariety of $\mathbb{R}^8$, those ambient metrics have holonomy equal to G_2.

In this dissertation, we extend these results to all generic 2-plane fields on 5-manifolds (equivalently, all Nurowski conformal structures). The following theorem, the first major application in this work, appears later as Theorem 3.1.2.

Theorem. *Suppose $\mathbb{D}$ is a real-analytic, generic 2-plane field an orientable 5-manifold, let $c_{\mathbb{D}}$ be the Nurowski conformal structure it naturally induces, and let $\widetilde{g}_{\mathbb{D}}$ be a real-analytic ambient metric of $c_{\mathbb{D}}$. Then, (possibly replacing $\widetilde{g}_{\mathbb{D}}$ with its restriction to some dilation-invariant open subset of $\widetilde{M}$ containing $\mathcal{G}$) $\mathrm{Hol}(\widetilde{g}_{\mathbb{D}}) \leq \mathrm{G}_2$.*

We apply a covering space argument to produce an analogous result for nonorientable manifolds.

After establishing this containment, we convert Leistner and Nurowski's technical criteria into pointwise polynomial conditions on a high-order jet of $\mathbb{D}$ (or more precisely, invoking the quasi-normal form, of F) that must hold if the holonomy of the ambient metric is a proper subgroup of G_2, and we show that the criterion is only satisfied on (a subset of) a proper subvariety of the appropriate jet space at that point. This yields this following genericity result, given here as Theorem 3.1.7. It roughly says that in the space of real-analytic, generic 2-plane fields $\mathbb{D}$ on oriented 5-manifolds, those whose Nurowski conformal structures have ambient holonomy equal to G_2 are dense.

Theorem. *There is a dense subset $S \subset J^7_0$ with the following property: If $F(x, y, p, q, z)$ is a real-analytic function whose 7-jet $j^7_0 F$ lies in S (and for which F_{qq} is nonvanishing) and $\widetilde{g}_F$ is a real-analytic ambient metric of the Nurowski conformal structure induced by $\mathbb{D}_F$, then* $\mathrm{Hol}(\widetilde{g}_F) = \mathrm{G}_2$.

In general, producing metrics with holonomy G_2 is difficult; this theorem shows that generic 2-plane fields on orientable 5-manifolds furnish an infinite-dimensional family of examples of such metrics. Moreover, the polynomial conditions exploited in the proof are practical in the sense that they can be tractably exploited to construct large families of metrics with holonomy G_2.

We prove the holonomy containment result using tractor geometry, which essentially captures the zeroth-order information of the ambient construction. Given a signature-(p, q) conformal structure c on an n-manifold M, one can form the **tractor bundle** $\mathcal{T} \to M$, which we define as the canonical rank-$(n+2)$ vector bundle whose sections are the sections of $T\widetilde{M}|_{\mathcal{G}}$ homogeneous of degree -1 with respect to the natural $\mathbb{R}^+$-action on that space [ČG03]. Then the restriction of any ambient metric induces a natural (fiber) *tractor metric* $g^{\mathcal{T}}$ on $\mathcal{T}$ and a *tractor connection* $\nabla^{\mathcal{T}}$. (These objects are independent of the choice of ambient metric.) This tractor connection is equivalent to the normal conformal connection of c, whose central role in conformal geometry is analogous to that of the Levi-Civita connection in pseudo-Riemannian geometry.

Just as for pseudo-Riemannian manifolds vis-à-vis the Levi-Civita connection, some geometric structures on conformal manifolds can be described by objects parallel with respect to the tractor connection. For example, nonzero parallel sections of the tractor bundle $\mathcal{T}$ of a conformal structure (M, c) (equivalently, the dual **cotractor bundle**, $\mathcal{T}^*$, which we may identify with $\mathcal{T}$ via $g^{\mathcal{T}}$) correspond to the so-called *almost Einstein scales* of the conformal structure. This notion modestly generalizes the notion of Einstein scales of a conformal structure, which by definition are in bijective correspondence with the Einstein representatives of c. (In fact, the existence of an [almost] Einstein representative of a given conformal structure is equivalent to the existence of a global solution to a certain second-order partial differential equation on (M, g), where $g \in c$ is any representative metric; then, one can characterize the tractor bundle as the subbundle of the 2-jet bundle over M defined pointwise by that equation [BEG94].) Extending attention to parallel sections of **tractor tensor bundles**, that is, subbundles of tensor powers of $\mathcal{T}^*$ (possibly with some indices raised), yields other structures. Hammerl and Sagerschnig showed [HS09] the fact, recorded here as Theorem 3.1.1, that a signature-$(2,3)$ conformal structure on an orientable manifold admits a parallel section of $\Lambda^3\mathcal{T}^*$ suitably compatible with the tractor metric (and necessarily of so-called *split* algebraic type) iff it is induced by a generic 2-plane field on a 5-manifold, that is, iff it is a Nurowski conformal structure.

One can realize G_2 as the stabilizer of a 3-form of split algebraic type on a 7-dimensional real vector space, so to show that the holonomy of a metric is contained in G_2, it suffices to show that it admits a parallel 3-form of that type. We have realized the tractor bundle as a restriction of the ambient bundle, so to show that ambient metrics of Nurowski conformal structures always have holonomy contained in G_2, it suffices to show that the parallel sections of $\Lambda^3\mathcal{T}^*$ produced by Hammerl and Sagerschnig extend to parallel 3-forms on the ambient manifold.

This extension problem led to the development of a much more general result, Theorem 2.1.2, which applies to all conformal structures of dimension $n \geq 3$. It guarantees that any parallel section of a tractor tensor bundle extends to a section of the corresponding bundle over the ambient space that is parallel at least to a certain explicit order; as for the ambient metric itself, the parity of the dimension plays a critical role. In general, call a tensor $\widetilde{\chi}$ on the ambient manifold an **ambient extension** of a tractor tensor χ if $\widetilde{\chi}|_{\mathcal{G}} = \chi$ and if $\widetilde{\chi}$ is homogeneous with respect to the natural dilations of the ambient bundle.

Theorem (Parallel Tractor Extension Theorem)**.** *Let (M, c) be a conformal manifold of dimension $n \geq 3$, and let $\widetilde{g}$ be an ambient metric for c.*

- *If n is odd, any parallel tractor tensor χ admits an ambient extension $\widetilde{\chi}$ satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^\infty)$.*

- *If n is even, any parallel tractor tensor χ admits an ambient extension $\widetilde{\chi}$ satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2-1})$.*

Gover proved this previously for the special case of rank-1 tensors on odd-dimensional manifolds [Gov10].

This theorem specializes in the case of odd n and real-analytic c to the following, Theorem 2.1.3. (In particular, these hypotheses apply to the Nurowski conformal structures induced by real-analytic plane fields $\mathbb{D}$.)

Theorem (Parallel Tractor Extension Theorem: odd, real-analytic case)**.** *Let (M, c) be a real-analytic conformal manifold of odd dimension, and let $(\widetilde{M}, \widetilde{g})$ be a real-analytic ambient manifold for c. Any parallel tractor tensor χ admits a unique and real-analytic parallel ambient extension $\widetilde{\chi}$ to an open neighborhood of $\mathcal{G}$ in $\widetilde{M}$.*

Chapter 1 comprises a detailed review of background material. Section 1.1 gives definitions and essential results for general Cartan and parabolic geometries and devotes some special attention to the Cartan curvature of (normal) conformal geometry. Section 1.2 recalls some basic facts about pseudo-Riemannian geometry and conformal geometry and reviews the standard metric bundle of a conformal structure, conformally weighted vector bundles, and related constructions. It then presents the Fefferman-Graham ambient construction for a conformal structure, and in particular recalls several facts about the curvature of an ambient metric. Section 1.3 constructs the tractor bundle $\mathcal{T}$ of a conformal structure c from the ambient metric and develops the splitting of $\mathcal{T}$ induced by a choice of representative metric of c, which will be used extensively, and then translates several constructions into the dual language of cotractors, which for convenience will be predominantly used thenceforth. It then recalls briefly the classical definition of the tractor bundle in terms of the conformally Einstein condition and shows that the two formulations agree, and also extends some of the constructions of the preceding sections to more general tractor tensor bundles. Section 1.4 constructs the groups G_2 and P and produces explicit representations of both of their Lie algebras using the nonassociative algebra of *split octonions*, and it derives some of their essential properties. Using the intrinsic geometry of that algebra, it describes explicitly the model of the geometry of generic 2-plane fields on (oriented) 5-manifolds, or equivalently, via the Čap-Schichl-Tanaka Theorem (recorded here as Theorem 1.1.20), parabolic geometries of type $(\mathfrak{g}_2, P)$ satisfying some normalization criteria. Section 1.5 constructs several objects naturally induced by a generic 2-plane field on a 5-manifold and describes the local quasi-normal form for those fields that arises from ordinary differential equations of form $z' = F(x, y, y', y'', z)$, which will be essential in the proof of genericity of G_2 holonomy for the induced ambient metrics. It then describes Nurowski's construction of a canonical conformal structure from such a plane field $\mathbb{D}$ in terms of parabolic geometry, and it constructs the fundamental curvature tensor for the geometry of those fields, which can be regarded as a section $A \in \Gamma(S^4\mathbb{D}^*)$, in terms of the Weyl curvature of the induced conformal structure. Section 1.6 defines and collects some key results about holonomy, including the local de Rham decomposition of a metric into a product of a flat metric and metrics with indecomposably acting holonomy, as well as Berger's List. It also collects several results special to holonomy contained in G_2, including the technical lemma devised by Leistner and Nurowski in their construction of metrics with holonomy equal to G_2, describes those metrics, and sketches a proof of that equality.

Chapter 2 presents the Parallel Tractor Extension Theorem, which describes the existence of parallel ambient extensions of parallel tractors on a general conformal manifold. Any Einstein representative of a conformal structure induces an canonical, bona fide Ricci-flat ambient metric, and in this setting any parallel tractor tensor admits a parallel extension for that metric (irrespective of real-analyticity and parity of dimension). Moreover, because the canonical metric is explicit in this case, we can compute any such extension explicitly. The extension of the parallel cotractor associated to the Einstein representative itself extends to a 1-form parallel with respect to the ambient metric, necessarily reducing the holonomy of the ambient metric to a proper subgroup of $\mathrm{O}(p+1, q+1)$. Section 2.2 investigates the case of even n, in which the existence of parallel extensions is not guaranteed beyond a certain critical order, and produces explicit conditions under which parallel tractor tensors of certain types admit extensions parallel beyond that order for at most one choice of ambient metric for the underlying conformal structure.

Chapter 3 describes some applications of the Parallel Tractor Extension Theorem. Section 3.1 shows that the holonomy of the real-analytic ambient metric of the conformal structure induced by a real-analytic 2-plane field $\mathbb{D}$ on an oriented 5-manifold is contained in G_2. It then shows, in the sense described above, that equality holds generically (Theorem 3.1.7) by producing the mentioned polynomial conditions on the fundamental curvature tensor A of the plane field and the Cotton and Weyl tensors of the induced conformal structure $c_{\mathbb{D}}$. It describes, too, additional structure enjoyed by such an ambient metric in the special case in which $c_{\mathbb{D}}$ is Einstein. Finally, Section 3.2 returns to the setting of general conformal structures and uses the extension theorem to produce a large family of necessary integrability conditions that a parallel tractor tensor of a given arbitrary type must satisfy. Finally, Section 3.3 suggests some possible directions for further extensions of the results in this work from the point of view of producing metrics of special holonomy.

0.1.1 Notation

All objects in hypotheses are assumed to be smooth unless stated otherwise (in particular, the loops that occur in the definitions of various notions of holonomy are assumed merely to be piecewise-smooth), and all manifolds are assumed to be connected and have dimension $n \geq 3$. For a manifold M we will sometimes denote spaces of sections of bundles over M as follows:

$$\begin{aligned} \mathcal{E} &:= C^\infty(M) \\ \mathcal{E}^a &:= \Gamma(TM) \\ \mathcal{E}_{(a_1 \cdots a_k)} &:= \Gamma(S^k T^* M) \\ \mathcal{E}_{[a_1 \cdots a_k]} &:= \Gamma(\Lambda^k T^* M). \end{aligned}$$

In particular, we will sometimes denote $\Gamma(T^*M)$ by $\mathcal{E}_a$. We extend this notation throughout this work. The notation $A \leq B$ indicates that A is a subgroup of B and the notation $\mathfrak{a} \leq \mathfrak{b}$ that $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{b}$. The symbols $<$, $>$, and $\geq$ are defined analogously.

Chapter 1

BACKGROUND

1.1 *Parabolic geometry*

A Cartan geometry is a manifold endowed with data realizing it as a curved version of a homogeneous space, generalizing the realization of Riemannian manifolds as curved versions of Euclidean space. Roughly speaking, one attaches to each point of the manifold a copy of the homogeneous space, and the additional data specifies how to connect the copies attached to different points. In the special case that the homogeneous space is the quotient of a semisimple Lie group by a parabolic subgroup, a Cartan geometry admits some special algebraic features, and the Cartan geometry is called a *parabolic geometry*. In this section, we largely follow [Čap06], [ČS09b], and [Sha97].

1.1.1 Klein geometries

Any Klein geometry, which is essentially a homogeneous space, defines a type of Cartan geometry, and any Cartan geometry of that type may be regarded locally as a deformation of that Klein geometry.

Definition 1.1.1. A **Klein geometry** (**homogeneous model**) is a pair (G, H) where G is a Lie group and $H \leq G$ is a closed (hence Lie) subgroup such that the quotient manifold G/H is connected. The group G is the **principal group** and the quotient manifold G/H is the **space** of the geometry.

By construction, the space of a Klein geometry (G, H) is a homogeneous space: G acts transitively and smoothly on G/H on the left by $a \cdot (gH) = (ag)H$. We may equivalently regard a Klein geometry as a smooth right principal H-bundle $G \to G/H$, where the bundle projection is the natural quotient map $G \to G/H$.

Two Klein geometries (G, H) and (G', H') are **geometrically isomorphic** if there is a

Lie group isomorphism $\varphi : G \to G'$ such that $\varphi(H) = H'$; by construction, φ is a bundle map between the principal bundles of the two Klein geometries defined as in the last paragraph.

To specify a Klein geometry, we may also give a principal group G and a connected space X on which G acts transitively, in which case H is the stabilizer of an (arbitrary) point in X. Different choices of points yield conjugate stabilizers H and hence, taking φ to be an appropriate conjugation, geometrically isomorphic Klein geometries.

The total space G of the bundle of a Klein geometry (G, H) admits a canonical $\mathfrak{g}$-valued 1-form (where $\mathfrak{g}$ is the Lie algebra of G): By pushing forward by left multiplication and applying the canonical identification $T_{\mathrm{id}}G \leftrightarrow \mathfrak{g}$, the form identifies with $\mathfrak{g}$ the tangent space to each point of the group. This in turn canonically identifies the tangent spaces at all points of G, and this form serves as a model for the data in a Cartan geometry that specifies how tangent spaces are connected.

Definition 1.1.2. The **Maurer-Cartan form** of a Lie group G is the form $\omega_{MC} \in \Gamma(T^*G \otimes \mathfrak{g})$ defined for $X \in T_gG$ by

$$\omega_{MC}(X) := dL_{g^{-1}}X. \tag{1.1}$$

The Maurer-Cartan form induces a canonical trivialization $TG \to G \times \mathfrak{g}$ via the identification

$$X \to (\pi_{TG}(X), \omega_{MC}(X)), \tag{1.2}$$

where $\pi_{TG} : TG \to G$ is the bundle projection.

1.1.2 Cartan geometries

Cartan geometries are curved versions of Klein geometries: Any Cartan geometry is infinitesimally equivalent at each point to the Klein geometry (G, H) on which it is modeled, but may be locally inequivalent (that is, curved); this curvature, which is encoded as a tensor that generalizes the Riemannian curvature, is a consequence of how nearby tangent spaces in the geometry are identified, and this is in turn specified by the Cartan connection of the geometry. By definition, Cartan connections generalize the Maurer-Cartan form on the total space of the bundle $G \to G/H$.

Definition 1.1.3. Given a Lie algebra $\mathfrak{g}$ and a Lie group H with Lie algebra $\mathfrak{h} \leq \mathfrak{g}$, a **Cartan connection** of type $(\mathfrak{g}, H)$ on a right principal H-bundle $\pi : E \to M$ is a $\mathfrak{g}$-valued 1-form $\omega : TE \to \mathfrak{g}$ satisfying the following axioms:

1. (Ad-invariance) For all $h \in H$,

$$R_h^* \omega = \mathrm{Ad}(h^{-1}) \circ \omega,$$

   where R_h is the right action map (for h), and $\mathrm{Ad} : H \to \mathrm{GL}(\mathfrak{g})$ is a fixed representation extending $\mathrm{Ad}_{\mathfrak{h}} : H \to \mathrm{GL}(\mathfrak{h})$

2. (Reproduction of the fundamental vector fields) For all $Y \in \mathfrak{h} \cong T_{\mathrm{id}}H$, $\omega(\zeta_Y) = Y$, where $\zeta_Y \in \Gamma(TE)$ is the **fundamental vector field** corresponding to Y: Let $h : \mathbb{R} \to H$ be the unique 1-parameter subgroup with initial tangent vector Y; we define $\zeta_Y := \left.\dfrac{d}{dt}\right|_0 R_{h(t)}$.

3. (Absolute parallelism) For all $e \in E$, ω restricts to an isomorphism $\omega|_e \colon T_eE \to \mathfrak{g}$.

The absolute parallelism condition ensures that the Cartan connection ω identifies the tangent space at each point of the total space E with $\mathfrak{g}$ and thus the tangent spaces at different points with one another, just as the Maurer-Cartan form on G identifies the tangent spaces of that group, and hence defines a natural trivialization $TE \to E \times \mathfrak{g}$ by

$$X \mapsto (\pi_{TE}(X), \omega(X)), \tag{1.3}$$

where $\pi_{TE} : TE \to E$ is the bundle projection.

A Cartan geometry is all of this structure taken together. (Note that, unlike in the definition of a Klein geometry, the definition of a Cartan geometry refers only to the Lie algebra $\mathfrak{g}$ of the larger group G and so disregards the global topology of that group.)

Definition 1.1.4. Given a Lie algebra $\mathfrak{g}$ and a Lie group H with Lie algebra $\mathfrak{h} \leq \mathfrak{g}$, a **Cartan geometry** of type $(\mathfrak{g}, H)$ is a pair $(E \to M, \omega)$, where $E \to M$ is a right principal H-bundle, and ω is Cartan connection of type $(\mathfrak{g}, H)$ on $E \to M$. A Cartan geometry of type $(\mathfrak{g}, H)$ is also called a $(\mathfrak{g}, H)$**-geometry** or a $(\mathfrak{g}, H)$**-structure** on the base manifold M.

One may define the category of Cartan geometries of type $(\mathfrak{g}, H)$ by declaring a morphism between two such geometries, $(E \to M, \omega)$ and $(E' \to M', \omega')$, to be a principal bundle morphism $\Phi : E \to E'$ that respects the Cartan connection in the sense that $\Phi^*\omega' = \omega$. Given any morphism Φ in this category, the absolute parallelism condition in the definition of Cartan connection forces the tangent map $T\Phi$ to be an isomorphism, and so both Φ and its base map are local diffeomorphisms. The geometries $(E \to M, \omega)$ and $(E' \to M', \omega')$ are **equivalent** (that is, isomorphic in this category) if there is a principal bundle morphism Φ that is a (global) diffeomorphism (or equivalently, if its base map is a global diffeomorphism). They are instead **locally equivalent** near $x \in M$ and $x' \in M'$ if there are neighborhoods U of x and U' of x' such that $(E|_U \to U, \omega|_U)$ and $(E'_{U'} \to U', \omega|_{U'})$ are equivalent.

After the Cartan connection ω, the most important tensorial data associated to a Cartan geometry $(E \to M, \omega)$ of type $(\mathfrak{g}, H)$ is its **curvature**, $K \in \Gamma(\Lambda^2 T^* E \otimes \mathfrak{g})$, the $\mathfrak{g}$-valued 2-form defined by the structural equation

$$K := d\omega + \omega \wedge \omega. \tag{1.4}$$

Since ω trivializes TE by realizing it isomorphically as $E \times \mathfrak{g}$, K (in fact, any differential form on E) is determined by its values on the **constant vector fields** $\omega^{-1}(X)$, $X \in \mathfrak{g}$. Thus, we can equivalently describe the curvature by the map $\kappa : E \to \Lambda^2 \mathfrak{g}^* \otimes \mathfrak{g}$ defined by

$$\kappa(u)(X, Y) := K(\omega^{-1}(X)(u), \omega^{-1}(Y)(u)); \tag{1.5}$$

applying the invariant formula for the exterior derivative to this definition gives $\kappa(u)(X, Y) = [X, Y] - \omega([\omega^{-1}(X), \omega^{-1}(Y)])(u)$. If $X \in \mathfrak{h}$, the constant vector field $\omega^{-1}(X)$ is the fundamental vector field ζ_X by definition of the Cartan connection. The equivariance of ω implies that $\mathcal{L}_{\zeta_X}\omega = \zeta_X \lrcorner\, d\omega = -\operatorname{ad}(X) \circ \omega$, and thus for all $\eta \in \Gamma(TE)$,

$$\begin{aligned} K(\omega^{-1}(X), \eta) &= d\omega(\omega^{-1}(X), \eta) + [\omega(\omega^{-1}(X)), \omega(\eta)] \\ &= -\operatorname{ad}(X)(\omega(\eta)) + [X, \omega(\eta)] \\ &= 0. \end{aligned}$$

Since η is arbitrary and the fundamental vector fields span the vertical bundle, K is horizontal (that is, it is annihilated by any vertical vector field). Thus, by construction, κ descends through the quotient to a map $E \to \Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{g}$ (also denoted κ).

A Cartan geometry is called **flat** if K (equivalently, κ) is identically zero. A Cartan geometry is called **torsion-free** if K takes its values in the subalgebra $\mathfrak{h}$, the Lie algebra of the group H (equivalently, if κ takes its values in $\Lambda^2(\mathfrak{g}/\mathfrak{h})^* \otimes \mathfrak{h}$). So, a Cartan geometry is torsion-free iff the image of κ under $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ is trivial.

Example 1.1.5 (Model geometry). The Klein geometry (G, H) endowed with the Maurer-Cartan form ω_{MC} of G can be regarded as the Cartan geometry $(G \to G/H, \omega_{MC})$ of type $(\mathfrak{g}, H)$, where $\mathfrak{g}$ is the Lie algebra of G. If G realizes $\mathfrak{g}$, we call $(G \to G/H, \omega_{MC})$ a **model geometry** of type $(\mathfrak{g}, H)$, and its base is called the **model space** (of that type); by construction, the model space is determined up to universal cover. The Maurer-Cartan identity

$$d\omega_{MC} + \omega_{MC} \wedge \omega_{MC} = 0 \tag{1.6}$$

then simply asserts that the model geometry is flat, so such a geometry is also called a **flat model**.

The group G acts transitively on G/H and preserves ω_{MC}, so a model geometry is locally equivalent to itself near any points $gH, g'H$ (via $L_{g'g^{-1}}$). So, if a Cartan geometry $(E \to M, \omega)$ of type $(\mathfrak{g}, H)$ is locally equivalent to the model geometry near $x \in M$ and $gH \in G/H$, then it is locally equivalent near x to the model geometry at every point. We say that $(E \to M, \omega)$ is **locally flat** if for every $x \in M$ it is locally equivalent near x to a model geometry at any (equivalently, every) point; pulling back to the universal cover of any model space shows that this characterization is independent of the choice of model geometry. In this sense, we may (locally) regard any other geometry of type $(\mathfrak{g}, H)$ as a deformation of the model geometry. One can show that the curvature of a Cartan geometry vanishes identically if and only if it is locally flat, so we may regard the curvature as an (again, local) measure of that deviation from the flat model.

Several familiar geometric structures can be realized as (that is, are essentially equivalent to) Cartan geometries for specific choices of the pair $(\mathfrak{g}, H)$. The first three examples below are discussed thoroughly in [Sha97].

Example 1.1.6 (Riemannian geometry). A torsion-free $(\mathfrak{ao}(n), \mathrm{O}(n))$-structure contains the same information as a **Riemannian structure**, that is, a Riemannian metric, on the base n-manifold. Here, $\mathrm{AO}(n) = \mathbb{R}^n \rtimes \mathrm{O}(n)$ is the group of isometries of $\mathbb{R}^n$ and $\mathfrak{ao}(n)$ is its Lie algebra. A flat model of the geometry is the space $\mathrm{AO}(n)/\mathrm{O}(n) \cong \mathbb{R}^n$ endowed with the standard Euclidean metric.

Example 1.1.7 (Conformal geometry). There is a canonical bijection between $(\mathfrak{o}(p+1, q+1), \dot{P})$-geometries satisfying a condition called *normality* (see the discussion of this geometry in Example 1.1.21) and **conformal structures** of signature (p, q), that is, equivalence classes of signature-(p, q) pseudo-Riemannian metrics on manifolds M, where two metrics are in the same class iff one is a positive, $C^\infty(M)$ multiple of the other. Here, $\dot{P}$ is the stabilizer in $\mathrm{O}(p+1, q+1)$ of a null ray in $\mathbb{R}^{p+1,q+1}$. One flat model of the geometry is the space of null rays in $\mathbb{R}^{p+1,q+1}$ endowed with the conformal structure inherited from the standard pseudo-Riemannian metric on $\mathbb{R}^{p+1,q+1}$; we may identify this structure with the product $\mathbb{S}^p \times \mathbb{S}^q$ endowed with the conformal class containing the product $g_p \oplus (-g_q)$ of the round metrics g_p and g_q on $\mathbb{S}^p$ and $\mathbb{S}^q$, respectively. (See Example 1.1.21 and Subsection 1.2.4 for further details of parabolic geometry of this type.)

Example 1.1.8 (Projective geometry). One may canonically associate to each normal $(\mathfrak{sl}(n+1, \mathbb{R}), S)$-structure a **torsion-free projective structure** on the base manifold, that is, a **projective equivalence class** of torsion-free linear connections on the base, where two connections are in the same class iff they determine the same unparameterized geodesics. Here, S is the stabilizer of a point in $\mathbb{RP}^n$ under the standard action of $\mathrm{SL}(n+1, \mathbb{R})$ on that space. This association, however, is not a bijection in the way that the analogous associations are for most other types of parabolic geometries; see the discussion of this geometry in Example 1.1.22 below. One flat model of the geometry is just $\mathbb{RP}^n$ endowed with the standard projective equivalence class (that is, the one containing the Levi-Civita connection induced on $\mathbb{RP}^n$ by realizing it as the quotient $\mathbb{S}^n/\mathbb{Z}_2$ of $\mathbb{S}^n$ with the round metric, where the nonidentity element in $\mathbb{Z}_2$ acts antipodally).

Example 1.1.9. One can also realize oriented analogs of Riemannian and conformal structures as Cartan geometries. For example, one may identify torsion-free $(\mathfrak{aso}(n), SO(n))$-geometries with Riemannian metrics on oriented manifolds, where $\mathfrak{aso}(n)$ is the Lie algebra of the group $\mathrm{ASO}(n) = \mathbb{R}^n \rtimes \mathrm{SO}(n)$ of orientation-preserving isometries of $\mathbb{R}^n$; the model space is $\mathrm{ASO}(n)/\mathrm{SO}(n) \cong \mathbb{R}^n$.

For any Lie group G containing H and realizing $\mathfrak{g}$, one may canonically extend the Cartan connection ω to a connection on a principal G-bundle canonically defined in terms of the given bundle $E \to M$, taking the map Ad to be just $\mathrm{Ad}_{\mathfrak{g}} : G \to G$. To do this, one forms the principal G-bundle $\overline{E} = E \times_H G$, on which ω extends equivariantly (and hence uniquely) to a principal connection $\overline{\omega}$. One may hence define the holonomy of the Cartan geometry $(E \to M, \omega)$ to be the principal bundle holonomy of the extended connection $\overline{\omega}$. (See Subsection 1.6.2.)

1.1.3 Parabolic geometries

Parabolic geometries are the special class of Cartan geometries $(\mathfrak{g}, P)$ for which $\mathfrak{g}$ is semisimple and the Lie algebra $\mathfrak{p}$ of P is a *parabolic subalgebra* of $\mathfrak{g}$.

We define parabolic subalgebras, then parabolic subgroups, and finally parabolic geometries, in terms of gradings and filtrations on $\mathfrak{g}$. One can endow semisimple Lie algebras with gradings (and thus filtrations) that arise from their root decompositions, and so these gradings and filtrations inherit in suitable ways the compatibility of those decompositions with the Lie bracket. Then, on a parabolic geometry $(E \to M, \omega)$, the corresponding Lie algebra grading induces a filtration on the tangent bundle of the base M via the Cartan connection ω.

Definition 1.1.10. A $|k|$**-grading** of a semisimple Lie algebra $\mathfrak{g}$ (where k is a positive integer) is a direct sum decomposition $\mathfrak{g} = \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_k$ such that

1. For all indices i and j, $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ (taking $\mathfrak{g}_i = \{0\}$ for $|i| > k$).

2. The subalgebra $\mathfrak{g}_- := \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1}$ is Lie-generated by the summand $\mathfrak{g}_{-1}$. (Condition (i) guarantees that $\mathfrak{g}_-$ is indeed a subalgebra.)

3. The summands $\mathfrak{g}_{-k}$ and $\mathfrak{g}_k$ are both nonzero.

Note that any form α (in particular, the curvature κ) that takes its values in a $|k|$-graded semisimple Lie algebra $(\mathfrak{g}_i)$ can be decomposed uniquely as a sum

$$\sum_{i=-k}^{k} \alpha_i \tag{1.7}$$

of forms α_i taking respective values in $\mathfrak{g}_i$.

Definition 1.1.11. The **parabolic subalgebra** associated to the $|k|$-grading $(\mathfrak{g}_i)$ of a semisimple Lie algebra $\mathfrak{g}$ is

$$\mathfrak{p} := \mathfrak{g}_0 \oplus \cdots \oplus \mathfrak{g}_k. \tag{1.8}$$

Condition (i) in Definition 1.1.10 guarantees that a parabolic subalgebra is in fact a subalgebra. Similarly, the subspace $\mathfrak{n} := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ induced by the grading is both a subalgebra of $\mathfrak{g}$ and a nilpotent ideal of $\mathfrak{p}$.

Like any grading of a vector space, a $|k|$-grading $(\mathfrak{g}_i)$ induces a canonical filtration

$$\mathfrak{g} = \mathfrak{g}^{-k} \supset \cdots \supset \mathfrak{g}^{k}, \tag{1.9}$$

where $\mathfrak{g}^i = \mathfrak{g}_i \oplus \cdots \oplus \mathfrak{g}_k$. Since the $|k|$-grading respects the Lie bracket in the sense of condition (i) of Definition 1.1.10, so does the filtration, that is, $[\mathfrak{g}^i, \mathfrak{g}^j] \subseteq \mathfrak{g}^{i+j}$ for all indices i, j. Note that, by definition, $\mathfrak{n} = \mathfrak{g}^1$.

Definition 1.1.12. A **parabolic subgroup** of a semisimple Lie group G realizing the $|k|$-graded Lie algebra $\mathfrak{g}$ is a subgroup $P \leq G$ which is contained in

$$\{g \in G : (\forall i)\ \mathrm{Ad}(g)(\mathfrak{g}^i) \subseteq \mathfrak{g}^i\} \tag{1.10}$$

and contains the connected component of the identity of that group.

Definition 1.1.13. The **Levi subgroup** of a semisimple Lie group G corresponding to the $|k|$-graded Lie algebra $\mathfrak{g}$ and a choice of parabolic subgroup $P \leq G$ is

$$G_0 := \{g \in G : (\forall i)\ \mathrm{Ad}(g)(\mathfrak{g}_i) \subseteq \mathfrak{g}_i\} \cap P. \tag{1.11}$$

By definition, G_0 preserves the $|k|$-grading $(\mathfrak{g}_i)$ on $\mathfrak{g}$ whereas P preserves the induced filtration $(\mathfrak{g}^i)$ (a weaker condition). The subgroup $N = \exp \mathfrak{n}$ (which is contained in P) is nilpotent, and one can show that $P = G_0 \rtimes N$.

We are now prepared to define a parabolic geometry:

Definition 1.1.14. Given a $|k|$-graded semisimple Lie algebra $\mathfrak{g}$ and a parabolic subgroup P of a Lie group G realizing $\mathfrak{g}$, a Cartan geometry of type $(\mathfrak{g}, P)$ is called a **parabolic geometry** (of type $(\mathfrak{g}, P)$).

Via the Cartan connection on a parabolic geometry, the defining $|k|$-grading induces filtrations on various bundles over M, including E; we introduce some notation and language for working with them.

Definition 1.1.15. A **filtered manifold** is a manifold S endowed with a filtration

$$TS = T^{-m}S \supseteq \cdots \supseteq T^{-1}S.$$

Such a filtration naturally induces a graded bundle $\mathrm{gr}(TS)$ with summands

$$\mathrm{gr}_i(TS) := T^iS/T^{i+1}S$$

(we set T^0S to be the vector subbundle of TS containing only the zero vector at each point, and we define $T^iS = TS$ for $i < -m$); let

$$q_i : T^iS \to \mathrm{gr}_i(TS)$$

denote the natural quotient maps.

Suppose that a filtered manifold respects the Lie bracket $[\cdot, \cdot]$ in the sense that, for all $\xi \in \Gamma(T^iS)$ and all $\eta \in \Gamma(T^jS)$, $[\xi, \eta] \in \Gamma(T^{i+j}S)$, for all indices i, j. Since the filtration indices are all negative, we have $T^iS, T^jS \subseteq T^{i+j+1}S$ for all indices i, j, and thus the natural map $\Gamma(T^iS) \times \Gamma(T^jS) \to \Gamma(\mathrm{gr}_{i+j}(TS))$ defined by $(\xi, \eta) \mapsto q_{i+j}([\xi, \eta])$ is induced by a tensor map $T^iS \times T^jS \to \mathrm{gr}_{i+j}(TS)$. By construction, the value of the tensor depends only on the classes of ξ and η in $\mathrm{gr}_i(TS)$ and $\mathrm{gr}_j(TS)$, respectively, so the map descends to a tensor map $\mathrm{gr}_i(TS) \times \mathrm{gr}_j(TS) \to \mathrm{gr}_{i+j}(TS)$. These maps together define an antisymmetric tensor operator

$$\mathcal{L} : \mathrm{gr}(TS) \times \mathrm{gr}(TS) \to \mathrm{gr}(TS)$$

called the **Levi bracket** induced by the filtered manifold structure. By construction, the restriction of $\mathcal{L}$ to any point $x \in S$ defines a nilpotent, graded Lie algebra structure on $\mathrm{gr}_x(TS)$ called the **symbol algebra** of S at x.

Recall that the Cartan connection ω of a parabolic geometry $(E \to M, \omega)$ of type $(\mathfrak{g}, P)$ induces a trivialization $TE \cong E \times \mathfrak{g}$ (see equation (1.3)). Thus, the filtration $(\mathfrak{g}^i)$ associated to the parabolic subgroup P induces a filtration

$$T^{-k}E \supseteq \cdots \supseteq T^kE$$

of TE with parts

$$T^iE := \omega^{-1}(\mathfrak{g}^i).$$

Since the adjoint P-action preserves the filtration (T^iE) (because it preserves the filtration $\mathfrak{g}^i$), it induces a natural identification $TM = E \times_P (\mathfrak{g}/\mathfrak{p})$, and the subfiltration $T^{-k}E \supseteq \cdots \supseteq T^{-1}E$ induces a natural filtration

$$T^{-k}M \supseteq \cdots \supseteq T^{-1}M.$$

This filtration then realizes TM as a filtered manifold compatible with the Lie bracket, and so a parabolic geometry defines a Levi bracket $\mathcal{L}$ on the resulting graded bundle $\mathrm{gr}(TM)$.

Since the nilpotent subgroup $N < P$ acts freely on E by construction, the quotient $E_0 = E/N$ is a principal bundle over M with structure group $P/N = G_0$. The Cartan connection ω then induces a bundle map from E_0 to the frame bundle of $\mathrm{gr}(TM)$, defining a reduction of the structure group of that bundle to $\mathrm{Ad}(G_0)$. In particular, the Cartan connection induces a refined (again, canonical) identification of vector bundles, $\mathrm{gr}_i(TM) \cong E_0 \times_{G_0} \mathfrak{g}_i$. Taking these together defines a canonical isomorphism $\mathrm{gr}(TM) \cong E_0 \times_{G_0} \mathfrak{g}_-$, and each fiber $\mathrm{gr}(T_xM)$ is a nilpotent graded Lie algebra canonically isomorphic to $\mathfrak{g}_-$. Moreover, since the Lie bracket on $\mathfrak{g}$, and hence that on $\mathfrak{g}_-$, is G_0-invariant, it induces via the above identification a (tensorial) map

$$\{\cdot, \cdot\} : \mathrm{gr}(TM) \times \mathrm{gr}(TM) \to \mathrm{gr}(TM)$$

called the **algebraic bracket**. This bracket is generally distinct from the Levi bracket induced by the filtration, but the two agree in the cases of principal interest.

Definition 1.1.16. A parabolic geometry $(E \to M, \omega)$ is **regular** if the induced algebraic and Levi brackets are the same map.

Remark 1.1.17. For a regular parabolic geometry, the agreement of the two brackets induces at each point $x \in M$ an isomorphism (of graded Lie algebras) between the symbol algebra $\mathrm{gr}_x(TM)$ and $\mathfrak{g}_-$.

Finally, any parabolic geometry $(E \to M, \omega)$ of type $(\mathfrak{g}, P)$ admits a canonical **adjoint tractor bundle**, $\mathcal{A}M = E \times_P \mathfrak{g}$, where P acts via the adjoint action. Since the associated filtration $(\mathfrak{g}^i)$ is P-invariant, it induces a filtration

$$\mathcal{A}^{-k}M \supset \cdots \supset \mathcal{A}^k M$$

of smooth (constant-rank) subbundles, and each fiber is a filtered Lie algebra isomorphic to $(\mathfrak{g}^i)$. The Cartan connection yields a canonical identification $TM \leftrightarrow E \times_P (\mathfrak{g}/\mathfrak{p})$ and thus an identification

$$TM \leftrightarrow \mathcal{A}M/\mathcal{A}^0 M$$

So, the filtration $(\mathcal{A}^i M)$ induces a filtration $TM = T^{-k}M \supset \cdots \supset T^0 M = M \times \{0\}$ of TM under the identifications $T^i M = \mathcal{A}^i M/\mathcal{A}^0 M$, and since $\mathcal{A}M$ is a quotient of TE, $(T^i M)$ agrees with the previous filtration so denoted. The curvature function κ may be realized in a third way, namely, as an $\mathcal{A}M$-valued 2-form on M: Direct computation shows that the curvature function regarded as a map $\kappa : E \to \Lambda^2(\mathfrak{g}/\mathfrak{p})^* \otimes \mathfrak{g}$ is P-equivariant (with respect to the adjoint action), so it corresponds to a smooth section of the associated bundle, namely, $\Lambda^2 T^*M \otimes \mathcal{A}M$.

Remark 1.1.18. One can show that a parabolic geometry is regular iff

$$\kappa(T^i M, T^j M) \subset \mathcal{A}^{i+j+1}M.$$

In particular, a torsion-free geometry is necessarily regular, so regularity may be regarded as a condition that the torsion of the parabolic geometry not be too severe.

The Killing form on $\mathfrak{g}$ induces a duality between $\mathfrak{g}/\mathfrak{p}$ and $\mathfrak{n} = \mathfrak{g}^1$, so the identification $TM \cong E \times_P (\mathfrak{g}/\mathfrak{p})$ yields an natural isomorphism $T^*M \cong E \times_P \mathfrak{n} = \mathcal{A}^1 M$. So, each cotangent space T_x^*M is a nilpotent Lie algebra whose bracket is the restriction of the algebraic bracket of $\mathcal{A}M$.

A primary reason for studying parabolic geometries is that they realize varied geometric structures, both familiar and novel, in a common axiomatic framework. One would like to establish bijective correspondences between (1) parabolic geometries of a particular type and with a given base manifold and (2) the possible geometric structures of the corresponding type on that base, but without additional restrictions, parabolic geometries have too much freedom to admit such identifications. To resolve this disparity, and in particular to produce the desired bijections for most parabolic geometries, one restricts attention to parabolic geometries satisfying both the regularity condition above and one other technical condition.

To frame conveniently this second condition, one defines the homology of a parabolic geometry $(E \to M, \omega)$ of type $(\mathfrak{g}, P)$ induced by the homology of the underlying Lie algebra $\mathfrak{g}$: For each $l > 0$, define the (tensorial) boundary operator

$$\partial^* : \Lambda^l T^*M \otimes AM \to \Lambda^{l-1} T^*M \otimes AM$$

by

$$\begin{aligned} \partial^*((\alpha_1 \wedge \cdots \wedge \alpha_l) \otimes s) := \sum_{i=1}^{l} (-1)^i (\alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \alpha_l) \otimes \{\alpha_i, s\} \\ + \sum_{i<j} (-1)^{i+j} (\{\alpha_i, \alpha_j\} \wedge \alpha_1 \wedge \cdots \wedge \widehat{\alpha_i} \wedge \cdots \wedge \widehat{\alpha_j} \wedge \cdots \wedge \alpha_l) \otimes s, \quad (1.12) \end{aligned}$$

where $\alpha_r \in T^*M$ for $r = 1, \ldots, l$, $s \in AM$, and $\{\cdot, \cdot\}$ denotes the brackets induced by the algebraic bracket on both AM and $T^*M \cong A^1M$. Computing directly shows that $(\partial^*)^2 = 0$ and, restricting to each fiber, the quotients $(\ker \partial^*)/(\operatorname{im} \partial^*)$ are the pointwise Lie algebra homologies of T_x^*M with coefficients in A_xM, respectively.

Definition 1.1.19. A parabolic geometry $(E \to M, \omega)$ (or the Cartan connection ω) is **normal** if $\partial^* \kappa = 0$, regarding κ as an AM-valued 2-form.

The above construction began with a generic regular parabolic geometry $(E \to M, \omega)$ of type $(\mathfrak{g}, P)$ and produced a particular geometric structure on M: a filtered manifold whose symbol algebras are all isomorphic to $\mathfrak{g}_-$ together with a reduction $E_0 \to M$ of the graded bundle $\mathrm{gr}(TM)$ to the structure group $\mathrm{Ad}(G_0) \leq \mathrm{Aut}_{\mathrm{gr}}\,\mathfrak{g}_-$; here, $\mathrm{Aut}_{\mathrm{gr}}\,\mathfrak{g}_-$ denotes the group of automorphisms of $\mathfrak{g}_-$ that preserve the grading. A fundamental result in the theory of parabolic geometries is a theorem essentially due to Tanaka, but given here in a form due to Čap and Schichl, which states that this construction may be inverted and, for most types of parabolic geometry, uniquely so.

Theorem 1.1.20 (Čap-Schichl-Tanaka)**.** *[ČS09b] Let $\mathfrak{g}$ be a $|k|$-graded semisimple Lie algebra, (T^iM) a filtered manifold such that each symbol algebra is isomorphic to $\mathfrak{g}_-$, and $E_0 \to M$ a reduction of $\mathrm{gr}(TM)$ to structure group $\mathrm{Ad}(G_0) \leq \mathrm{Aut}_{\mathrm{gr}}\,\mathfrak{g}_-$. Then, there is a regular, normal parabolic geometry $(E \to M, \omega)$ of type $(\mathfrak{g}, P)$ for a suitable parabolic subgroup P that yields the given data via the construction in this section. Moreover, if $H_1(\mathfrak{n}, \mathfrak{g})$ is concentrated in nonpositive homogeneous degrees, then that parabolic geometry is unique up to isomorphism. (Here $H_1(\mathfrak{n}, \mathfrak{g})$ is the first Lie algebra homology group of $\mathfrak{n}$ with coefficients in $\mathfrak{g}$.)*

When working with a particular type of parabolic geometry, the somewhat abstract conditions of regularity and normality often reduce to more concrete ones. In particular, if $\mathfrak{g}$ is $|1|$-graded, parabolic geometries of a type $(\mathfrak{g}, P)$ are vacuously regular.

Parabolic geometries are numerous and diverse.

Example 1.1.21 (Conformal structures). An $(\mathfrak{o}(p+1, q+1), \dot{P})$-structure (see Example 1.1.7) is normal if the zeroth-graded summand κ_0 of its curvature (see equation (1.7)) is tracefree (equivalently, if it is formally Ricci-flat), that is, if $(\kappa_0)_{ki}{}^k{}_j = 0$. Cartan constructed a bijection between signature-(p, q) conformal structures and what are now called normal $(\mathfrak{o}(p+1, q+1), \dot{P})$-structures. See Subsection 1.2.4 for additional discussion of conformal geometry as parabolic geometry; in particular, it shows that the parabolic subgroup $\dot{P}$ corresponds to a $|1|$-grading of $\mathfrak{o}(p+1, q+1)$, and so parabolic geometries of this type are vacuously regular.

Example 1.1.22 (Projective structures). Any $(\mathfrak{sl}(n+1, \mathbb{R}), P)$-structure (again, cf. Example 1.1.8) yields a torsion-free projective structure. The parabolic P corresponds to a $|1|$-grading of $\mathfrak{sl}(n+1, \mathbb{R})$ and so is regular; the homology group $H_1(\mathfrak{n}, \mathfrak{sl}(n+1, \mathbb{R}))$ is not concentrated in nonpositive homogeneous degrees, though, so the Čap-Schichl-Tanaka Theorem does not uniquely determine a corresponding normal $(\mathfrak{sl}(n+1,\mathbb{R}),P)$-structure. When $n>1$, however, some other underlying data can be used to make a (unique) canonical choice among the normal $(\mathfrak{sl}(n+1,\mathbb{R}),P)$-geometries inducing a given torsion-free projective structure.

Remark 1.1.23. Besides projective structures, the only other types $(\mathfrak{g},P)$ of parabolic geometries with $\mathfrak{g}$ simple for which $H_1(\mathfrak{n},\mathfrak{g})$ is not concentrated in nonpositive homogeneous degrees are the so-called contact projective structures, for which $\mathfrak{g}=\mathfrak{sp}(2n,\mathbb{R})$ and P has a certain isomorphism type. Thus, for all other types of geometries with $\mathfrak{g}$ simple, the Čap-Schichl-Tanaka Theorem establishes a bijective correspondence between the remaining data of the hypotheses of that theorem (that is, the associated geometric structure) and normal, regular parabolic geometries of the corresponding type.

Example 1.1.24 (Generic 2-plane fields on 5-manifolds)*.* This is the example of principal interest in this work. Let G_2 be the (algebra) automorphism group of the split octonions $\widetilde{\mathbb{O}}$ and $P<\mathrm{G}_2$ the stabilizer of a null ray in $\operatorname{Im}\widetilde{\mathbb{O}}\cong\mathbb{R}^{3,4}$ under the induced natural action of G_2 on that vector space. Cartan essentially showed, albeit in different language, that there is a canonical bijection between regular, normal $(\mathfrak{g}_2,P)$-structures and oriented 2-plane fields $\mathbb{D}$ on 5-manifolds such that $[\mathbb{D},[\mathbb{D},\mathbb{D}]]=TM$ (see Definition 1.5.1, which defines the bracket $[A,B]$ of two plane fields) [Car10]. The parabolic subgroup P is induced by a $|3|$-grading of $\mathfrak{g}_2$, so the regularity condition is not vacuous for this type of geometry. A model space is $\mathbb{S}^2\times\mathbb{S}^3$, and one flat model is a G_2-invariant 2-plane field on this space. One can construct a nonoriented analogue of this geometry by replacing P instead with the stabilizer P' of a null ray in $\mathrm{G}_2\times\mathbb{Z}_2$. See Section 1.4 for explicit constructions of the groups G_2 and P, and Section 1.5 for details of this geometric structure.

Example 1.1.25 (Generic 3-plane fields on 6-manifolds)*.* [Bry06] Bryant showed that there is a canonical bijection between regular, normal $(\mathfrak{so}(3,4),Q)$-structures and 3-plane fields $\mathbb{E}$ on a 6-manifold that satisfy $[\mathbb{E},\mathbb{E}]=TM$. By construction, $\mathrm{SO}(3,4)$ acts naturally on the 6-dimensional projective subvariety $\mathrm{Gr}_0(3,\mathbb{R}^{3,4})\subset\mathrm{Gr}(3,\mathbb{R}^{3,4})$ comprising the null 3-planes in $\mathbb{R}^{3,4}$; here, Q is the stabilizer of a null 3-plane under this action. The model space is $\mathrm{Gr}_0(3,\mathbb{R}^{3,4})$, and one flat model is a nondegenerate, $\mathrm{SO}(3,4)$-invariant 3-plane field on this space. This type of geometry is formally similar to the 2-plane type discussed in the previous example, and so it admits analogs of many of the results in Section 1.5. See Section 3.3 in this dissertation for further discussion.

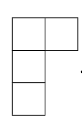
In this dissertation, we will always discuss parabolic geometries of these types in the context of the corresponding underlying geometries, and so via Theorem 1.1.20 we will always take them by hypothesis to be normal and regular.

1.2 Conformal and ambient geometry

1.2.1 *Young diagrams and irreducible* $\mathrm{GL}(\mathbb{V})$*- and* $\mathrm{O}(h)$*-representations*

In this section we review a simple diagrammatic system useful for (among numerous applications) parameterizing the irreducible representations of the general linear and orthogonal groups. We mostly follow Fulton and Harris [FH, Subsections 4.1, 15.3, 19.5], who present the constructions only for vector spaces over $\mathbb{C}$, though they apply unchanged to vector spaces over any field of characteristic 0. Henceforth, all representations are finite-dimensional by hypothesis.

For a fixed nonnegative integer r, a **partition** λ of r is a sequence $(r_1, \ldots, r_s)$ of positive integers such that $\sum_k r_k = r$ and $r_1 \geq \cdots \geq r_s$. (So, the only partition of 0 is the empty sequence $\emptyset$.) We can depict a partition as a **Young diagram**, a collection of top-justified columns of boxes with exactly r_k boxes in the kth leftmost column. For example, the partition $(3, 1)$ of 4 corresponds to the Young diagram


.

(Young diagrams are often instead specified by giving instead the number of boxes in each row rather than in each column. In that setting, the partition $(r_1, \ldots, r_s)$ as specified above is the called **conjugate partition** of the corresponding Young diagram.) By convention, the unique partition $\emptyset$ of 0 is assigned the Young diagram with zero boxes, which we denote $\bullet$.

A **Young tableau** is a Young diagram, say, corresponding to the partition $\lambda = (r_1, \ldots, r_s)$ of r, together with a bijective assignment, called a **filling**, of the numbers $1, \ldots, r$ to the boxes of the diagram. For example, one filling of the above Young diagram is

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array}\ .$$

The symmetric group S_r then acts transitively on the set of fillings of the diagram. Fix a filling and define $P \leq S_r$ to be the subgroup comprising all the permutations that for each row preserve the set of numbers assigned to the boxes in that row, and define $Q \leq S_r$ to be the subgroup that similarly preserves each column. Let $\mathbb{F}$ be a field of characteristic 0, and in the group algebra $\mathbb{F}S_r$, define the elements

$$a_\lambda := \sum_{g \in P} e_g$$

$$b_\lambda := \sum_{g \in Q} \operatorname{sgn}(g) e_g.$$

For any vector space $\mathbb{V}$ over $\mathbb{F}$, S_r acts on $\bigotimes^r \mathbb{V}$ by permutation of factors. Suppose that the row lengths of the diagram are $t_1 \geq \cdots \geq t_u$. By construction, the induced (right) action of a_λ on $\bigotimes^r \mathbb{V}$ has image equal to

$$S^{t_1}\mathbb{V} \otimes \cdots \otimes S^{t_u}\mathbb{V} \subseteq \bigotimes^r \mathbb{V},$$

where the inclusion is given by grouping factors by row, and the action of b_λ has image

$$\Lambda^{r_1}\mathbb{V} \otimes \cdots \otimes \Lambda^{r_s}\mathbb{V} \subseteq \bigotimes^r \mathbb{V},$$

where the inclusion is given by grouping factors by column. We define the **Young symmetrizer** of the tableau to be the group element

$$c_\lambda := a_\lambda \cdot b_\lambda \in \mathbb{F}S_r.$$

Finally, we let $\mathbb{S}_\lambda$ denote the corresponding **Schur functor**, which sends $\mathbb{V}$ to the image of $c_\lambda \in \operatorname{End}(\bigotimes^r \mathbb{V})$,

$$\mathbb{S}_\lambda \mathbb{V} := \left(\bigotimes^r \mathbb{V}\right).c_\lambda.$$

By construction, this image is a $\mathrm{GL}(\mathbb{V})$-subrepresentation of $\bigotimes^r \mathbb{V}$, and any tensor in this space has the symmetries preserved by c_λ. Different choices of Young tableaux with the same underlying Young diagram yield different groups P and Q and different elements a, b, and c of $\operatorname{End}(\bigotimes^r \mathbb{V})$, but give isomorphic representations $\mathbb{S}_\lambda \mathbb{V}$.

Theorem 1.2.1. *Suppose $\mathbb{V}$ is a vector space of dimension n over a field of characteristic 0. Then, the map $\lambda \mapsto \mathbb{S}_\lambda \mathbb{V}$ defines a bijective correspondence between the partitions for which $r_1 \leq n$ (equivalently, the Young diagrams with at most n rows) and (isomorphism classes of) the irreducible representations of* $\mathrm{GL}(\mathbb{V})$.

In particular, we may indicate any irreducible representation of $\mathrm{GL}(\mathbb{V})$ (up to isomorphism) just by giving the corresponding Young diagram. By construction,

$$\mathbb{S}_\lambda \mathbb{V} \subseteq S^{d_n} \Lambda^n \mathbb{V} \otimes \cdots \otimes S^{d_2} \Lambda^2 \mathbb{V} \otimes S^{d_1} \mathbb{V} \subseteq \Lambda^{r_1} \mathbb{V} \otimes \cdots \otimes \Lambda^{r_s} \mathbb{V},$$

where d_k is the number of times the positive integer k occurs in the partition λ.

Now, let $\mathbb{V}$ be as above and let $h \in S^2 \mathbb{V}^*$ be a nondegenerate, symmetric bilinear form. In general, the restriction of an irreducible $\mathrm{GL}(\mathbb{V})$-representation to the orthogonal group $\mathrm{O}(h)$ of h is no longer irreducible, so we describe how to modify the above constructions to yield an analogous parameterization of the irreducible representations of $\mathrm{O}(h)$.

Fix r; for each pair (p, q) of integers such that $1 \leq p < q \leq r$, define the linear **contraction** map

$$\Phi_{(p,q)} : \bigotimes^r \mathbb{V} \to \bigotimes^{r-2} \mathbb{V}$$

by

$$\Phi_{(p,q)} : v_1 \otimes \cdots \otimes v_r \mapsto h(v_p, v_q) v_1 \otimes \cdots \otimes v_{p-1} \otimes v_{p+1} \cdots \otimes v_{q-1} \otimes v_{q+1} \otimes \cdots \otimes v_r,$$

and denote the common kernel of these maps by

$$\mathbb{V}^{[r]} := \bigcap_{1 \leq p < q \leq r} \ker \Phi_{(p,q)}.$$

For any partition (equivalently, Young diagram), the intersection

$$\mathbb{S}_{[\lambda]} \mathbb{V} := \mathbb{V}^{[r]} \cap \mathbb{S}_\lambda \mathbb{V}$$

is a representation of $\mathrm{O}(h)$, and we call any tensor in this space **(totally) tracefree**.

Theorem 1.2.2. *Suppose $\mathbb{V}$ is a vector space of dimension n over a field of characteristic 0. The map $\lambda \mapsto \mathbb{S}_{[\lambda]} \mathbb{V}$ defines a bijective correspondence between the partitions for which $r_1 + r_2 \leq n$ (equivalently, the Young diagrams whose first and second columns together contain at most n boxes) and the irreducible representations of* $\mathrm{O}(h)$.

We indicate the representation $\mathbb{S}_{[\lambda]}\mathbb{V}$ of $\mathrm{O}(h)$ by giving the corresponding Young diagram and marking it with a subscript 0. In the context of $\mathrm{O}(h)$-representations, a Young diagram without the subscript 0 just indicates the restriction of the $\mathrm{GL}(\mathbb{V})$-representation $\mathbb{S}_\lambda\mathbb{V}$ to $\mathrm{O}(h)$, which again need not be irreducible.

Example 1.2.3. The zero-box Young diagram $\bullet$ (equivalently, the empty partition $\emptyset$) corresponds to the trivial representation.

Example 1.2.4. The $\mathrm{O}(h)$-representation $\underbrace{\square\cdots\square}_{r}{}_0$ (equivalently, the representation induced by the partition $(1,\dots,1)$ of r) is just the space $S^r_0\mathbb{V}$ of tracefree, symmetric r-tensors.

Example 1.2.5. The $\mathrm{O}(h)$-representation ${}_r\left\{\begin{matrix}\square\\ \vdots\\ \square\end{matrix}\right.$ (equivalently, the representation induced by the partition (r) of r) is just the space $\Lambda^r\mathbb{V}$ of totally antisymmetric r-tensors. Now, $\mathbb{S}_{[(r)]}\mathbb{V} = \mathbb{S}_{(r)}\mathbb{V}$, so these representations are irreducible.

Henceforth, we always take $\mathbb{V} = (\mathbb{R}^m)^*$ for appropriate m.

1.2.2 Geometric preliminaries

Given a pseudo-Riemannian metric g of signature (p, q), the Levi-Civita connection ∇ of g is the unique torsion-free connection preserving the metric, that is, satisfying $g_{ab,c} = 0$. The Riemannian curvature R of g is the 4-tensor field defined by

$$R^d{}_{abc}\eta_d := 2\eta_{a,[bc]}$$

for all $\eta_a \in \mathcal{E}_a$, and takes values in the representation $\ydiagram{2,2}$, so it satisfies the symmetries $R_{abcd} = -R_{bacd} = -R_{abdc} = R_{cdab}$ and $R_{a[bcd]} = 0$. As an $O(p, q)$-representation, it decomposes as

$$\ydiagram{2,2} \cong \ydiagram{2,2}_{\,0} \oplus \ydiagram{2}.$$

Now, $\ydiagram{2,2}_{\,0}$ is the representation comprising the tensors S satisfying the symmetries of $\ydiagram{2,2}$ and also the tracefree condition $S_{acb}{}^c = 0$, and $\ydiagram{2}$ is the space S^2 of symmetric 2-tensors. The space S^2 is embedded in $\ydiagram{2,2}$ via the inclusion

$$A \mapsto g \owedge A,$$

where $\owedge$ is the Kulkarni-Nomizu product of symmetric 2-tensors defined by

$$(A \owedge B)_{abcd} := 2A_{c[a}B_{b]d} + 2A_{d[b}B_{a]c}.$$

Then, the projections of the curvature tensor R onto the $\yng(2,2)_0$ and $\yng(2)$ components are, respectively,

$$\begin{aligned} R_{abcd} &\mapsto W_{abcd} := (R - g \owedge P)_{abcd} \\ R_{abcd} &\mapsto (g \owedge P)_{abcd}. \end{aligned}$$

Here, W is called the **Weyl tensor**, and P is the **Schouten tensor**, which is a trace adjustment of the **Ricci tensor**, $R_{ab} := R_{acb}{}^{c}$; P is characterized by

$$R_{ab} := (n-2)P_{ab} + P_{d}{}^{d} g_{ab}. \tag{1.13}$$

Also important in conformal geometry is the **Cotton tensor**,

$$C_{abc} := P_{ab,c} - P_{ac,b}.$$

By construction it takes values in the representation

$$\young(ba,c)_{\,0},$$

where the labeling indicates that it satisfies $C_{abc} = -C_{acb}$, $C_{[abc]} = 0$, and $C_{ab}{}^{a} = 0$. For $n > 3$, it is determined by the Weyl tensor via the divergence identity $W_{abcd,}{}^{a} = (3-n)C_{bcd}$. Finally, the **Bach tensor** is defined by

$$B_{ab} := C_{abc,}{}^{c} - P^{cd}W_{cabd}$$

and takes values in the representation $\yng(2)_0$, that is, it satisfies $B_{ab} = B_{ba}$ and $B_{a}{}^{a} = 0$.

1.2.3 Conformal geometry and the metric bundle

Given a manifold M, a **conformal structure** (or **conformal class**) c is an equivalence class of pseudo-Riemannian metrics on M, where $g \sim \widehat{g}$ if there is some positive function $\Omega \in \mathcal{E}$ such that $\widehat{g} = \Omega^2 g$. The pair (M, c) is called a **conformal manifold**. Any choice

$g \in c$ is called a **representative metric** of c. By construction, every metric in a conformal class has the same signature and so we define the signature of a conformal class to be the signature of an arbitrary representative $g \in c$.

One can encode a conformal structure (of arbitrary signature) on a manifold by constructing a bundle whose fibers comprise the nondegenerate, symmetric, bilinear forms on the respective tangent spaces determined by the conformal structure. (In this subsection we partially follow [FG, Section 2].)

Definition 1.2.6. The **metric bundle** of a conformal structure (M, c) is the ray bundle

$$\pi : \mathcal{G} \to M$$

whose fiber $\mathcal{G}_x$ is the ray of pseudo-inner products yielded by restricting representatives of the conformal class to T_xM, that is,

$$\mathcal{G}_x := \{g|_{T_xM} \colon g \in c\}.$$

By definition, the (global) sections of $\mathcal{G}$ are exactly the representative metrics in the conformal class, that is, there is a canonical identification $c \leftrightarrow \Gamma(\mathcal{G})$. A choice of representative $g \in c$ trivializes $\mathcal{G}$ as the product $M \times \mathbb{R}^+$, where the pair (x, t) is identified with $t^2 g_x \in \mathcal{G}$.

The **dilations** $\delta^s : \mathcal{G} \to \mathcal{G}$, $s \in \mathbb{R}^+$, defined by

$$\delta^s(x, g) := (x, s^2 g).$$

endow $\mathcal{G}$ with a principal $\mathbb{R}^+$-structure, via the action $(x, g) \cdot s = \delta^s(x, g)$. The infinitesimal generator of the dilations δ^s is the natural vector field

$$\mathbb{T} := \tfrac{d}{ds}\big|_1 \delta^s \in \Gamma(T\mathcal{G}). \tag{1.14}$$

Via the trivialization $\mathcal{G} \leftrightarrow M \times \mathbb{R}^+$ induced by the choice $g \in c$, $\mathbb{T} = t\partial_t$.

The metric bundle can also be viewed as a natural setting for managing information about homogeneity of conformal data on M. To this end, we use $\mathcal{G}$ to define another class of bundles:

Definition 1.2.7. For a **weight** $w \in \mathbb{R}$, the bundle of **conformal densities of weight** w is the bundle $\mathcal{D}[w] \to M$ with fiber

$$\mathcal{D}[w]_x := \{f : \mathcal{G}_x \to \mathbb{R} : (\delta^s)^* f = s^w f, s \in \mathbb{R}^+\}.$$

Trivially, any section $f \in \Gamma(\mathcal{D}[0])$ is constant on the fibers of $\mathcal{G}$, so we may identify $\mathcal{D}[0]$ with $M \times \mathbb{R}$. By construction, $\mathcal{D}[w] \otimes \mathcal{D}[w']$ is naturally isomorphic to $\mathcal{D}[w + w']$.

We can construct **conformally weighted vector bundles** by forming tensor products with the bundles $\mathcal{D}[w]$: For any real vector bundle V, denote

$$V[w] := V \otimes \mathcal{D}[w].$$

By construction, sections of $\mathcal{D}[w]$ are just real-valued functions on $\mathcal{G}$ homogeneous of degree w. So, via the usual identification of sections of associated bundles with equivariant vector-valued functions on principal bundles, we may identify $\mathcal{D}[w]$ with the vector bundle $\mathcal{G} \times_{\mathbb{R}^+} \mathbb{R}$ induced by the representation $\mathbb{R}^+ \to \mathrm{GL}(\mathbb{R})$ defined by $s \cdot t := s^{-w} t$. This identification suggests the alternate notation $\Gamma(V)[w]$ for the space $\Gamma(V[w])$ of sections of $V[w]$. Then, using the $\mathcal{E}$ notation for spaces of sections of certain tensor bundles, we may for example compactly denote $C^\infty(M)[w] = \Gamma(D[w])$ by $\mathcal{E}[w]$ and $\Gamma(TM[w])$ by $\mathcal{E}^a[w]$.

Regarding a choice $g \in c$ as a section $M \to \mathcal{G}$ identifies sections of weighted conformal density bundles with functions: If $f \in \mathcal{E}[w]$, then $f \circ g \in \mathcal{E}$. By construction, another representative $\widehat{g} = \Omega^2 g$ then identifies f with $f \circ \widehat{g} = \Omega^w f \circ g$.

The metric bundle $\mathcal{G}$ also admits a tautological symmetric, bilinear form $\mathbf{g} \in S^2 T^* \mathcal{G}$, defined for $X, Y \in T_{(x,h)} \mathcal{G}$ by

$$\mathbf{g}(X, Y) := h(d\pi_{(x,h)}(X), d\pi_{(x,h)}(Y)).$$

We may naturally identify $\mathbf{g}$ with the conformal structure itself, and by homogeneity it is a section in $\mathcal{E}_{(ab)}[2]$. Via the identification $\mathcal{G} \leftrightarrow M \times \mathbb{R}^+$ induced by a choice of representative $g \in c$, $\mathbf{g}$ is identified with $t^2 g$, where we suppress the notation π^* for the pullback of a tensor on M to $\mathcal{G}$.

1.2.4 Conformal geometry as parabolic geometry of type $(\mathfrak{o}(p+1,q+1),\dot{P})$

In this subsection we elaborate on Examples 1.1.7 and 1.1.21 and collect more facts about conformal geometry regarded as parabolic geometry.

In a basis in which a nondegenerate, symmetric, bilinear form on $\mathbb{R}^{p+1,q+1}$ has the form

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & h_{ab} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \tag{1.15}$$

the Lie algebra $\mathfrak{o}(p+1,q+1)$ of the orthogonal group preserving the bilinear form has representation

$$\left\{ \begin{pmatrix} \lambda & Z_b & 0 \\ X^a & Y^a_{\ b} & -Z^a \\ 0 & -X_b & -\lambda \end{pmatrix} : \lambda \in \mathbb{R}, X^a \in \mathbb{R}^{p,q}, Y^a_{\ b} \in \mathfrak{o}(h), Z_b \in (\mathbb{R}^{p,q})^* \right\},$$

where indices are raised and lowered with h. Computing brackets shows that decomposition $\mathfrak{o}(h) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ indicated by the labeling

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_1 & 0 \\ \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_{-1} \\ 0 & \mathfrak{g}_{-1} & \mathfrak{g}_0 \end{pmatrix} \tag{1.16}$$

is a $|1|$-grading on $\mathfrak{o}(h)$. If we take $\dot{P}$ to be the stabilizer of the null ray spanned by the first basis vector, its Lie subalgebra coincides with the parabolic algebra corresponding to the $|1|$-grading,

$$\dot{\mathfrak{p}} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \left\{ \begin{pmatrix} \lambda & Z_b & 0 \\ 0 & Y^a_{\ b} & -Z^a \\ 0 & 0 & -\lambda \end{pmatrix} : \lambda \in \mathbb{R}, Y^a_{\ b} \in \mathfrak{o}(h), Z_b \in (\mathbb{R}^{p,q})^* \right\}.$$

We can identify this subalgebra with $(\mathbb{R}^{p,q})^* \rtimes \mathfrak{co}(h)$, where $\mathfrak{co}(h)$ is the Lie algebra of the (nonoriented) conformal group $\mathrm{CO}(h) := \mathrm{O}(h) \cdot \mathbb{R}^+$. The corresponding nilpotent subalgebra is

$$\dot{\mathfrak{n}} = \mathfrak{g}_1 = \left\{ \begin{pmatrix} 0 & Z_b & 0 \\ 0 & 0 & -Z^a \\ 0 & 0 & 0 \end{pmatrix} : Z_b \in (\mathbb{R}^{p,q})^* \right\},$$

which we may identify with the abelian Lie algebra $(\mathbb{R}^{p,q})^*$.

For any conformal structure c, let $(\dot{E}, \dot{\omega})$ be the corresponding normal parabolic geometry of type $(\mathfrak{o}(p+1, q+1), \dot{P})$, and let $\dot{\sigma} \in \Gamma(\dot{E} \to M)$ be an arbitrary section. By construction, $\dot{\sigma}$ determines a representative metric $g \in c$: One can realize E as a principal bundle over the metric bundle $\mathcal{G}$ of c, and so there is a unique metric g, regarded as a section of $\mathcal{G}$, such that $\dot{\sigma}$ factors as $\dot{\sigma} = \psi \circ g$ for some section $\psi \in \Gamma(\dot{E} \to \mathcal{G})$. Then, the pullback of the curvature $\dot{\Omega}$ of $\dot{\omega}$ along an arbitrary section $\dot{\sigma} \in \Gamma(\dot{E} \to M)$ is an $\mathfrak{o}(p+1, q+1)$-valued 2-form on M and is given (suppressing the pullback notation) by

$$\dot{\Omega}_{abCD} = \begin{pmatrix} 0 & C_{dab} & 0 \\ 0 & W_{ab}{}^{c}{}_{d} & -C^{c}{}_{ab} \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.17}$$

where W and C are respectively the Weyl and Cotton tensors of the induced representative metric g [ČS09b, Section 1.6]. Consulting (1.16) shows that we may identify the zeroth-graded summand κ_0 of the curvature κ with the Weyl curvature.

1.2.5 The Fefferman-Graham ambient construction

The Fefferman-Graham ambient construction encodes a signature-(p, q) conformal structure c on a $(p+q)$-manifold M in a signature-$(p+1, q+1)$ pseudo-Riemannian *ambient metric* that is unique in a suitable sense. In this section we largely follow [FG].

The tautological 2-tensor $\mathbf{g}$ (see Subsection 1.2.3) on the metric bundle $\mathcal{G}$ associated to a signature-(p, q) conformal structure c on a manifold M is degenerate. One can thicken $\mathcal{G}$ to a $(n+2)$-manifold $\mathcal{G} \times \mathbb{R}$, however, and then construct an *ambient metric* $\widetilde{g}$ of signature $(p+1, q+1)$ on that space that pulls back to $\mathbf{g}$ in a suitable sense, satisfies certain defining conditions, and is defined on a suitable open subset $\widetilde{M} \subset \mathcal{G} \times \mathbb{R}$. Then, one can apply the tools of pseudo-Riemannian geometry to such an extended metric to produce information about the original conformal structure.

Let ρ denote the coordinate on the $\mathbb{R}$ factor of $\mathcal{G} \times \mathbb{R}$. We identify the metric bundle, $\mathcal{G}$, with $\mathcal{G} \times \{0\} \subset \mathcal{G} \times \mathbb{R}$ via the inclusion $\iota : z \mapsto (z, 0)$, and in this context $\mathcal{G}$ is called the **initial surface**. Define for $s \in \mathbb{R}^+$ the dilations $\delta^s : \mathcal{G} \times \mathbb{R} \to \mathcal{G} \times \mathbb{R}$ by $\delta^s(g_x, \rho) = (s^2 g_x, \rho)$, which hence define a $\mathbb{R}^+$-action on $\mathcal{G} \times \mathbb{R}$, and denote the infinitesimal generator of δ^s by $\mathbb{T} = \frac{\partial}{\partial s}|_1 \delta^s$. By construction δ^s and $\mathbb{T}$ extend the so-named maps and vector field on $\mathcal{G}$. Subsection 1.2.3 showed that choosing a representative $g \in c$ yields a global trivialization of $\mathcal{G}$ by identifying $(x,t) \in M \times \mathbb{R}^+$ with $(x, t^2 g_x) \in \mathcal{G}$. Using this trivialization, we may give points in $\mathcal{G} \times \mathbb{R}$ as triples $(t, x, \rho) \in \mathbb{R}^+ \times M \times \mathbb{R}$, which defines an embedding $M \hookrightarrow \mathcal{G} \times \mathbb{R}$ by $x \to (1, x, 0)$; furthermore, δ^s is then given by $\delta^s(t, x, \rho) = (st, x, \rho)$, and $\mathbb{T}$ is just $t\partial_t$. We use uppercase Latin indices, $A, B, C, \ldots$, for objects on $\mathcal{G} \times \mathbb{R}$, and when we have chosen a representative $g \in c$, we use a 0 index for the metric bundle fiber ($\mathbb{R}^+$) factor, lowercase Latin indices, $a, b, c, \ldots$, for objects on the M factor, and an ∞ index for the $\mathbb{R}$ factor in the induced identification.

Call a metric $\widetilde{g}$ of signature $(p+1, q+1)$ on an open neighborhood $\widetilde{M}$ of $\mathcal{G}$ a **pre-ambient metric** for a conformal structure (M, c) if $\iota^* \widetilde{g} = \mathbf{g}$ and $(\delta^s)^* \widetilde{g} = s^2 \widetilde{g}$ for $s \in \mathbb{R}^+$ (in particular, the latter condition requires that $\widetilde{M}$ be invariant under dilation). A pre-ambient metric is **straight** if $\widetilde{\nabla}\mathbb{T} = \mathrm{id}_{T\widetilde{M}}$, where $\widetilde{\nabla}$ is the Levi-Civita connection of $\widetilde{g}$.

Conformal geometry is fundamentally different in even and odd dimensions, and this is especially clear from the viewpoint of ambient geometry: We define an ambient metric to be a pre-ambient metric that satisfies an additional condition, but this condition depends on the parity of the dimension. If n is odd, an **ambient metric** for a conformal structure (M, c) is a straight, pre-ambient metric for that structure that is Ricci-flat to infinite order along $\mathcal{G}$.

The appropriate formulation for even n is more subtle: If S_{AB} is a symmetric 2-tensor field on an open neighborhood of $\mathcal{G} \subset M \times \mathbb{R}$ and $m \geq 0$, denote $S_{AB} = O^+_{AB}(\rho^m)$ if

(i) $S_{AB} = O(\rho^m)$;

(ii) for each $z \in \mathcal{G}$, $(\iota^*(\rho^{-m} S))(z) = \pi^* s$ for some $g_{\pi(z)}$-tracefree $s \in S^2 T^*_{\pi(z)} M$. (Here, $\rho^{-m} S$ denotes the unique continuous extension of that tensor across $\mathcal{G}$.)

If n is even, an **ambient metric** for a conformal structure (M, c) is a straight, pre-ambient metric for that structure that satisfies $\mathrm{Ric}(\widetilde{g}) = O^+_{AB}(\rho^{n/2-1})$.

If $\widetilde{g}$ is an ambient metric on $\widetilde{M}$ for the conformal structure (M, c), then the pair $(\widetilde{M}, \widetilde{g})$ is called an **ambient manifold** for (M, c).

Theorem 1.2.8. *[FG] If (M, c) is a conformal manifold of dimension $n \geq 3$, there is an ambient metric $\widetilde{g}$, and it is unique in the following sense:*

- *If n is odd, it is unique up to infinite order and up to pullback by a diffeomorphism defined on a dilation-invariant neighborhood of $\mathcal{G}$ in $\mathcal{G} \times \mathbb{R}$ which commutes with the dilations δ^s and fixes $\mathcal{G}$ pointwise.*

- *If n is even, it is unique up to addition of terms satisfying $O^+_{AB}(\rho^{n/2})$ and up to pullback by a diffeomorphism defined on a dilation-invariant neighborhood of $\mathcal{G}$ in $\mathcal{G} \times \mathbb{R}$ which commutes with the dilations δ^s and fixes $\mathcal{G}$ pointwise.*

For conformal manifolds (M, c) of even dimension $n \geq 4$, the existence of ambient metrics Ricci-flat to order higher than $O(\rho^{n/2-1})$ is obstructed precisely by a conformally invariant tensor, the *ambient obstruction tensor*, $\mathcal{O}$. In fact, (M, c) admits an ambient metric Ricci-flat to infinite order iff $\mathcal{O} = 0$. For such n we define an **infinite-order ambient metric** for a conformal manifold (M, c) to be a straight pre-ambient metric $\widetilde{g}$ for (M, c) for which $\mathrm{Ric}(\widetilde{g})$ vanishes to infinite order. When n is even, any choice of infinite-order ambient metric is determined by the curvature component $\widetilde{R}_{\infty ab\infty,\underbrace{\infty\cdots\infty}_{n/2-2}}$, and this component may be freely prescribed to be an arbitrary tracefree, symmetric 2-tensor (on M) whose divergence with respect to the representative $g \in c$ defining the splitting is equal to a natural 1-form that depends on g.

If a given conformal class c contains a real-analytic metric, then it determines a real-analytic ambient metric unique up to the indicated order and up to diffeomorphism preserving $\mathcal{G}$ pointwise. In particular, if n is odd, then c determines a (real-analytic) ambient metric unique up to diffeomorphism fixing $\mathcal{G}$ pointwise and up to extension.

The diffeomorphism invariance of an ambient metric can be broken by putting it into a normal form with respect to a choice of representative metric $g \in c$. Say that a pre-ambient metric $\widetilde{g}$ on $\widetilde{M}$ is in **normal form** with respect to g if

(i) for each $z \in \mathcal{G}$, $I_z = \widetilde{M} \cap (\{z\} \times \mathbb{R})$ is an open interval;

(ii) for each $z \in \mathcal{G}$, I_z is a parameterized geodesic, with parameterization $\rho \mapsto (z, \rho)$;

(iii) under the identification $\mathcal{G} \times \mathbb{R} \cong \mathbb{R}^+ \times M \times \mathbb{R}$ induced by g, $\widetilde{g}|_{\mathcal{G}} = 2t\, dt\, d\rho + t^2 g$.

The following results specify exactly the uniqueness of ambient metrics in normal form.

Lemma 1.2.9. *A straight, pre-ambient metric $\widetilde{g}$ of a conformal structure (M, c) is in normal form with respect to a representative $g \in c$ iff it has the form*

$$\widetilde{g} = 2\rho\, dt^2 + 2t\, dt\, d\rho + t^2 g_\rho \tag{1.18}$$

(in terms of the identification $\mathcal{G} \times \mathbb{R} \cong \mathbb{R}^+ \times M \times \mathbb{R}$ induced by g), where g_ρ is a smooth family of metrics on M parameterized by ρ that satisfies $g_0 = g$.

For any pre-ambient metric $\widetilde{g}$ of a conformal structure (M, c) and a choice of representative $g \in c$, there is a unique diffeomorphism ϕ that commutes with the dilations δ^s, $s \in \mathbb{R}^+$, and fixes $\mathcal{G}$ pointwise such that $\phi^* \widetilde{g}$ is in normal form with respect to g.

Theorem 1.2.10. *[FG, Theorem 2.9] Let (M, c) be a conformal structure, and choose a conformal representative $g \in c$. There is an ambient metric in normal form with respect to g. If n is odd, g_ρ is determined to infinite order at $\rho = 0$. If n is even, g_ρ is determined uniquely modulo $O(\rho^{n/2})$, and $\operatorname{tr}_g(\partial_\rho^{n/2}|_0 g_\rho)$ is determined.*

In principle, one computes the coefficients of the power series $\sum_{k=0}^{\infty} \mu^{(k)} \rho^k$ of g_ρ inductively by analyzing the conditions imposed on them by requiring that the Ricci curvature vanish along the initial surface to increasing orders; in that sense, the Ricci curvature condition propagates the initial data g_0 off $\mathcal{G}$. The terms $\mu^{(k)}$ of the general solution have only been computed for the first several k. The first is

$$\mu^{(1)}_{ab} = 2P_{ab}. \tag{1.19}$$

Provided that $n > 4$, the second is

$$\mu^{(2)}_{ab} = -\frac{1}{4-n} B_{ab} + P_{ac} P^c{}_b. \tag{1.20}$$

The full sequence $(\mu^{(k)})$ is known only for a few special classes of conformal structures.

One case in which the full sequence $(\mu^{(k)})$ is known is for Einstein conformal classes. A metric is **Einstein** iff its Ricci curvature is a constant multiple of g, that is, it satisfies the condition $R_{ab} = 2\lambda(n-1)g_{ab}$ for some constant λ, or, equivalently, $P_{ab} = \lambda g_{ab}$. The constant λ, which is just $\frac{1}{n}P_d{}^d$, is called the **Einstein constant** of the metric. (Elsewhere, $2\lambda(n-1)$ is sometimes called the Einstein constant.) A conformal structure is **Einstein** if it contains an Einstein representative.

If c contains the Einstein representative $g \in c$, say, with Einstein constant λ, then c admits a distinguished ambient metric $\widetilde{g}^{\mathrm{can}}$; its normal form with respect to g is

$$\widetilde{g}^{\mathrm{can}}(t,x,\rho) := 2\rho\, dt^2 + 2t\, dt\, d\rho + t^2 g_\rho, \qquad g_\rho(x) = (1+\lambda\rho)^2 g(x). \tag{1.21}$$

Computing shows that it is Ricci-flat. One can show that every Einstein representative $g \in c$ induces the same ambient metric (1.21) up to diffeomorphism and infinite order.

1.2.6 Ambient curvature

We collect some facts here about the covariant derivative, $\widetilde{\nabla}$, and curvature tensor, $\widetilde{R}$, of an ambient metric $\widetilde{g}$.

Fix a conformal structure c. Computing directly gives that the Christoffel symbols, $\widetilde{\Gamma}^C_{AB}$, of an ambient metric in normal form (1.18) with respect to $g \in c$ are:

$$\widetilde{\Gamma}^0_{AB} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{1}{2}t g'_{ab} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.22}$$

$$\widetilde{\Gamma}^c_{AB} = \begin{pmatrix} 0 & t^{-1}\delta_b{}^c & 0 \\ t^{-1}\delta_a{}^c & \Gamma^c_{ab} & \frac{1}{2}g^{cd}g'_{ad} \\ 0 & \frac{1}{2}g^{cd}g'_{bd} & 0 \end{pmatrix} \tag{1.23}$$

$$\widetilde{\Gamma}^\infty_{AB} = \begin{pmatrix} 0 & 0 & t^{-1} \\ 0 & -g_{ab} + \rho g'_{ab} & 0 \\ t^{-1} & 0 & 0 \end{pmatrix}. \tag{1.24}$$

Everywhere, g_{ab} denotes $(g_\rho)_{ab}$, and $'$ denotes the derivative ∂_ρ.

We now compute the curvature $\widetilde{R}$ of an ambient metric $\widetilde{g}$ in normal form with respect to a representative $g \in c$, specializing the indices according to the splitting $A \leftrightarrow (0, a, \infty)$. Computing directly again using (1.18) gives

$$\begin{aligned}\widetilde{R}_{0BCD} &= 0\\ \widetilde{R}_{abcd} &= t^2\left[R_{abcd} - \tfrac{1}{2}(g \owedge g')_{abcd} + \tfrac{\rho}{2}(g'_{ac}g'_{bd} - g'_{ad}g'_{bc})\right]\\ \widetilde{R}_{\infty bcd} &= \tfrac{1}{2}t^2[(g'_{bc})_{,d} - (g'_{bd})_{,c}]\\ \widetilde{R}_{\infty bc\infty} &= \tfrac{1}{2}t^2[g''_{bc} - \tfrac{1}{2}g^{ef}g'_{be}g'_{cf}].\end{aligned}$$

Here, g represents g_ρ and R represents the curvature of that metric. Now, these expressions depend on the choice of the ambient metric; however, since the sequence $(\mu^{(k)})$ is determined by a given $g \in c$ (although just up to $k = \frac{n}{2} - 1$ for n even), and $g^{(k)} = k!\,\mu^{(k)}$, the restrictions $\widetilde{R}_{ABCD}|_{\mathcal{G}}$ are independent of that extension, except that $\widetilde{R}_{\infty bc\infty}$ depends on the choice of ambient metric of c for $n = 4$. Since $\widetilde{R}$ is homogeneous, no information is lost by restriction to $M \subset \widetilde{M}$, that is, to $\{t = 1, \rho = 0\}$. Using formulas (1.19) and (1.20) for, respectively, $\mu^{(1)}$ and $\mu^{(2)}$ gives that the ambient curvature along $M \subset \widetilde{M}$ is given by

$$\begin{aligned}\widetilde{R}_{0bcd}|_M &= 0\\ \widetilde{R}_{abcd}|_M &= W_{abcd}\\ \widetilde{R}_{\infty bcd}|_M &= C_{bcd}\\ \widetilde{R}_{\infty bc\infty}|_M &= -(n-4)^{-1}B_{bc}.\end{aligned} \tag{1.25}$$

(The formula for $\widetilde{R}_{\infty bc\infty}|_M$ holds only for $n \neq 4$.) Under changes of representative $g \in c$, tensors transform naturally under an action of $\mathrm{O}(p+1, q+1)$.

Similarly, the components of the restrictions $\widetilde{\nabla}^s\widetilde{R}|_M$ of the covariant derivatives of $\widetilde{R}$ to $M \subset \widetilde{M}$ are independent of the choice of ambient metric when n odd, and when n even and the general formula for the component does not contain derivatives of g of too high an order. The following gives a condition for n even for such a component to be independent of the choice of ambient metric.

Definition 1.2.11. The **strength** $||A||$ of a specialized index A in the splitting $A \leftrightarrow (0, a, \infty)$ is $||0|| := 0$, $||a|| := 1$, $||\infty|| := 2$. The strength of a multi-index $A_1 \cdots A_s$ of specialized indices A_u in the above splitting is $||A_1 \cdots A_s|| := \sum_{u=1}^{s} ||A_u||$.

Proposition 1.2.12. *[FG, Proposition 6.2] Suppose c is a conformal structure of even dimension $n \geq 4$. Choose a specialization of the multi-index $ABCDE_1 \cdots E_s$; if*

$$||ABCDE_1 \cdots E_s|| \leq n+1,$$

then the component $\widetilde{R}_{ABCD,E_1 \cdots E_s}|_M$ of $\widetilde{\nabla}^s \widetilde{R}$ is independent of the choice of ambient metric $\widetilde{g}$ of c.

The homogeneity of $\widetilde{g}$ induces a homogeneity of $\widetilde{R}$, which implies identities for components of the derivatives $\widetilde{\nabla}^s \widetilde{R}$ whose multi-indices contain a 0 index in the above splitting. By construction, with respect to the splitting induced by an arbitrary representative $g \in c$, $\mathbb{T}^0 = t$ and $\mathbb{T}^a = \mathbb{T}^\infty = 0$.

Proposition 1.2.13. *The covariant derivatives $\widetilde{\nabla}^s \widetilde{R}$ of a straight, pre-ambient metric $\widetilde{g}$ satisfy*

$$\mathbb{T}^D \widetilde{R}_{ABCD,E_1 \cdots E_s} = -\sum_{k=1}^{s} \widetilde{R}_{ABCE_k,E_1 \cdots E_{k-1} E_{k+1} \cdots E_s} \tag{1.26}$$

$$\begin{aligned} \mathbb{T}^F \widetilde{R}_{ABCD,E_1 \cdots E_i F E_{i+1} \cdots E_s} = -(i+2)\widetilde{R}_{ABCD,E_1 \cdots E_s} \\ - \sum_{k=i+1}^{s} \widetilde{R}_{ABCD,E_1 \cdots E_i E_k E_{i+1} \cdots E_{k-1} E_{k+1} \cdots E_s}. \end{aligned} \tag{1.27}$$

1.3 Conformal tractor geometry

Thomas showed one can naturally encode a conformal structure in a vector bundle over the underlying manifold of the structure, the *tractor bundle* [Tho26]. This bundle enjoys several interesting, equivalent formulations. We develop it in Subsection 1.3.1 below using the ambient construction; in Subsection 1.3.3 we show this formulation is equivalent to the standard one found in the seminal reference [BEG94], then go on to derive more needed properties of the bundle. The former perspective partially motivates the main theorem, Theorem 2.1.2, and the latter emphasizes the role of Einstein metrics in the tractor construction. Both perspectives emphasize a splitting of the bundle naturally induced by a choice of representative metric in the conformal class, which enables efficient and explicit computation; this splitting is described in detail in Subsection 1.3.2.

1.3.1 Ambient formulation of the tractor bundle

In this subsection we partly follow [ČG03].

Fix a conformal structure c of signature (p,q) and an ambient manifold $(\widetilde{M}, \widetilde{g})$ of c. The **(standard) tractor bundle** of c is the rank-$(n+2)$ bundle $\mathcal{T} \to M$, $n = p+q$, with fiber

$$\mathcal{T}_x := \{\chi \in \Gamma(T\widetilde{M}|_{\mathcal{G}_x}) : T\delta^s(\chi) = s\chi, s \in \mathbb{R}^+\}.$$

We call a section of $\mathcal{T}$ a **(standard) tractor** (or **tractor field**), and by definition we may canonically identify it with a section in $\Gamma(T\widetilde{M}|_{\mathcal{G}})$ homogeneous of degree -1. (Henceforth, homogeneity always refers to homogeneity with respect to pullback by the dilations δ^s.) Since $\widetilde{g}$ is homogeneous of degree 2, so is its restriction $\widetilde{g}|_{\mathcal{G}}$, and thus for any $\alpha, \beta \in \Gamma(\mathcal{T})$, $\widetilde{g}|_{\mathcal{G}}(\alpha, \beta)$ is homogeneous of degree 0, that is, constant on the fibers $\mathcal{G}_x$; hence, it defines a fiber metric $g^{\mathcal{T}}$ of signature $(p+1, q+1)$ on $\mathcal{T}$, the **tractor metric**. We call the dual bundle $\mathcal{T}^*$ of $\mathcal{T}$ the **cotractor bundle** and sections of it **cotractors** or **tractor 1-forms**. The tractor metric identifies $\mathcal{T}$ and $\mathcal{T}^*$.

We denote indices on $\mathcal{T}$ by Latin uppercase letters, $A, B, C, \ldots$, so that we may write a standard tractor χ as χ^A. In analogy with the $\mathcal{E}$ notation described in the introduction, we will sometimes denote $\Gamma(\mathcal{T})$ by $\mathcal{E}^A$ and $\Gamma(\mathcal{T}^*)$, by $\mathcal{E}_A$. We may form additional bundles by considering arbitrary subbundles of the tensor powers $\bigotimes^r \mathcal{T}^* \otimes \bigotimes^{r'} \mathcal{T}$. We call such bundles **tractor tensor bundles** and their sections **tractor tensors** (or **tractor tensor fields**). For example, the subbundle $\Lambda^r \mathcal{T}^* \subset \bigotimes^r \mathcal{T}^*$ of totally antisymmetric tractor r-tensors is a bundle whose sections we call **tractor r-forms**. Likewise, we call sections of the subbundle $S^r \mathcal{T}^* \subset \bigotimes^r \mathcal{T}^*$ of totally symmetric r-tensors simply **symmetric tractor r-tensors**. We further extend the $\mathcal{E}$ notation to include bundles of these types: $\mathcal{E}_{[A_1 \cdots A_r]}$ will denote $\Gamma(\Lambda^r \mathcal{T}^*)$, and $\mathcal{E}_{(A_1 \cdots A_r)}$ will denote $\Gamma(S^r \mathcal{T}^*)$. By definition, $g^{\mathcal{T}} \in \mathcal{E}_{(AB)}$. Raising an index of $\Lambda^2 \mathcal{T}$ yields a distinguished tractor tensor bundle, the **adjoint tractor bundle** $\mathcal{A}$, which we can also denote $\operatorname{End}_{\text{skew}}(\mathcal{T})$ and may identify with $\mathfrak{so}(\mathcal{T})$.

The tractor metric, together with the infinitesimal generator $\mathbb{T}$ of the dilations δ^s, naturally yields a vector bundle filtration

$$\mathcal{T}^1 \subset \mathcal{T}^0 \subset \mathcal{T}$$

of $\mathcal{T}$, which thus induces a natural graded bundle,

$$\mathrm{gr}(\mathcal{T}) := \mathcal{T}^1 \oplus (\mathcal{T}^0/\mathcal{T}^1) \oplus (\mathcal{T}/\mathcal{T}^0).$$

We define the pieces of the filtration and identify explicitly the components of this grading with objects on the base manifold.

Since $\mathbb{T}$ generates the dilations δ^s, they preserve the line subbundle $\mathrm{span}\{\mathbb{T}\} \subset T\widetilde{M}|_{\mathcal{G}}$, and we may furthermore regard $\mathrm{span}\{\mathbb{T}\}$ as a subbundle $\mathcal{T}^1 \subset \mathcal{T}$ (which is $g^{\mathcal{T}}$-null because $\mathbb{T}$ is $\widetilde{g}|_{\mathcal{G}}$-null). Now, sections of $\mathrm{span}\{\mathbb{T}\}$ are precisely the vector fields of the form $f\mathbb{T}$. Since $\mathbb{T}$ is homogeneous of degree 0, it lies in $\mathcal{E}^A[1]$. So if $f\mathbb{T} \in \Gamma(\mathcal{T})$, we must have $f \in \mathcal{E}[-1]$; then, the map $f \mapsto f\mathbb{T}$ defines a natural isomorphism

$$\mathcal{E}[-1] \cong \Gamma(\mathcal{T}^1),$$

and composing it with the inclusion $\mathcal{T}^1 \hookrightarrow \mathcal{T}$ gives a map $\iota : \mathcal{D}[-1] \hookrightarrow \mathcal{T}$. This map and the section map it induces, also denoted ι, are both called the **canonical injection** of the tractor bundle.

Since $\mathcal{T}^1$ is $g^{\mathcal{T}}$-null, it is contained in its (rank-$(n+1)$) orthogonal bundle $\mathcal{T}^0 := (\mathcal{T}^1)^{\perp}$, completing the definition of the filtration.

We may easily identify the quotient $\mathcal{T}/\mathcal{T}^0$: Define the map $\Pi_0 : \mathcal{T} \to \mathcal{D}[1]$ by $\Pi_0 : \chi^A \to \chi^A \mathbb{T}_A$. By definition of $\mathcal{T}^0$, Π_0 vanishes precisely on $\mathcal{T}^0$, so Π_0 descends to an isomorphism

$$\mathcal{T}/\mathcal{T}^0 \cong \mathcal{D}[1],$$

and thus we may regard it fiberwise just as the vector space reduction modulo $\mathcal{T}^0$. We call both Π_0 and the corresponding section map (also denoted Π_0) the **canonical projection** of the tractor bundle.

Finally, we identify the middle quotient, $\mathcal{T}^0/\mathcal{T}^1$ of the natural filtration, completing our description of the induced graded bundle $\mathrm{gr}(\mathcal{T})$. Consider any tractor $X \in \Gamma(\mathcal{T}^0)$, that is, a section of $\Gamma(T\mathcal{G})$ homogeneous of degree -1. Then, for any $f \in \mathcal{E}[1]$, fX is a vector field homogeneous of degree 0 tangent to $\mathcal{G}$, so it is a lift of a unique vector field $\widehat{X} \in \Gamma(TM)$ via the projection $\mathcal{G} \to M$, defining a map $\mathcal{T}^0 \to TM[-1]$. Since $\mathbb{T}$ spans the tangent space to the fibers of $\mathcal{G}$, the kernel of the map comprises exactly the tractors in $\mathcal{T}^1$, and so the map descends to an isomorphism

$$\mathcal{T}^0/\mathcal{T}^1 \cong TM[-1].$$

Rewriting this equation in terms of sections of $T\mathcal{G} \subset T\widetilde{M}|_{\mathcal{G}}$ and forming the tensor product of both sides with $\mathcal{D}[1]$ (to move the weight to the other side of the equation) yields a realization of the tangent bundle TM, with fiber

$$T_xM \leftrightarrow \{V \in \Gamma(T\mathcal{G}|_{\mathcal{G}_x}) : d\delta^s(V) = V, s > 0\}/\mathrm{span}\{\mathbb{T}\}.$$

We show that via this identification, the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{g}$ to $\mathcal{G}$ defines a natural connection $\nabla^{\mathcal{T}} : \mathcal{E}^A \times \mathcal{E}^b \to \mathcal{E}^C$. For an arbitrary point $x \in M$ and arbitrary sections $\chi \in \mathcal{E}^A$ and $Z \in \mathcal{E}^b$, pick a section $\bar{Z} \in \Gamma(T\mathcal{G}|_{\mathcal{G}_x})$ representing Z in the above realization; define

$$\nabla^{\mathcal{T}}_Z \chi := \widetilde{\nabla}_{\bar{Z}} \chi.$$

(On the right-hand side, χ is regarded as a section of $T\widetilde{M}|_{\mathcal{G}}$.) Though χ is only defined along $\mathcal{G}$, $\bar{Z}$ is everywhere tangent to $T\mathcal{G} \subset TM|_{\mathcal{G}}$, and so we can take the indicated covariant derivative. Any other lift of Z to $\Gamma(T\mathcal{G}|_{\mathcal{G}_x})$ has the form $\bar{Z} + f\mathbb{T}$ for some function $f \in \mathcal{E}$. Then,

$$\widetilde{\nabla}_{\bar{Z}+f\mathbb{T}} \chi = \widetilde{\nabla}_{\bar{Z}} \chi + f \widetilde{\nabla}_{\mathbb{T}} \chi.$$

Now, extend χ arbitrarily to a section of $T\widetilde{M}$, which we also denote χ, with the same homogeneity, and regard $\mathbb{T}$ as the restriction of the infinitesimal generator, which we also denote $\mathbb{T}$, of the dilations δ^s on $\widetilde{M}$. Then, on $\widetilde{M}$,

$$\widetilde{\nabla}_{\mathbb{T}} \chi = \widetilde{\nabla}_{\chi} \mathbb{T} + [\mathbb{T}, \chi] = \widetilde{\nabla}_{\chi} \mathbb{T} + \mathcal{L}_{\mathbb{T}} \chi;$$

since $\widetilde{g}$ is straight, $\widetilde{\nabla}_{\chi} \mathbb{T} = \chi$ on $\mathcal{G}$, and by homogeneity, $\mathcal{L}_{\mathbb{T}} \chi = -\chi$, so $\widetilde{\nabla}_{\mathbb{T}} \chi = 0$. Restricting back to $\mathcal{G}$ gives that $\widetilde{\nabla}_{\mathbb{T}} \chi = 0$, so $\widetilde{\nabla}_{\bar{Z}} \chi$ is independent of the lift $\bar{Z}$ of Z, and hence $\nabla^{\mathcal{T}}_Z \chi$ is well-defined. Moreover, counting homogeneities shows that it is a tractor. Finally, $\nabla^{\mathcal{T}}$ inherits linearity over $\mathcal{E}$ in the $\mathcal{E}^b$ argument and linearity over $\mathbb{R}$ in the $\mathcal{E}^A$ argument from $\widetilde{\nabla}$, so it defines a connection as desired. Since $\widetilde{\nabla} \widetilde{g} = 0$, we have $\nabla^{\mathcal{T}} g^{\mathcal{T}} = 0$.

This connection gives rise to a natural curvature.

Definition 1.3.1. The **tractor curvature** is the tractor tensor $R^{\mathcal{T}} \in \Gamma(\Lambda^2 T^*M \otimes \Lambda^2 \mathcal{T}^*)$ defined by

$$(R^{\mathcal{T}})_{ab}{}^{C}{}_{D}\chi^{D} := 2\chi^{C}{}_{,[ab]}$$

for all $\chi \in \mathcal{E}^A$.

As with any vector bundle connection, to define the curvature of $\nabla^{\mathcal{T}}$ we must specify the connection on M with which it couples so that we can define the second covariant derivative of a tractor. Let ∇ be the Levi-Civita connection of an arbitrary representative $g \in c$. Then, define the covariant derivative $\nabla^{\mathcal{T}} : \Gamma(\mathcal{T} \otimes T^*M) \to \Gamma(\mathcal{T} \otimes T^*M \otimes T^*M)$ of an arbitrary section $\alpha \otimes \eta \in (\mathcal{T} \otimes T^*M)$ by $\nabla^{\mathcal{T}}(\alpha \otimes \eta) = \nabla^{\mathcal{T}}\alpha \otimes \eta + \alpha \otimes \nabla\eta$. One can check that the definition of $R^{\mathcal{T}}$ does not depend on the choice of g.

Remark 1.3.2. We may view the tractor bundle as a special case of a much more general construction on a general parabolic geometry. For any conformal structure, let (E, ω) be the corresponding normal parabolic geometry of type $(\mathfrak{o}(p+1, q+1), \dot{P})$. For this choice of parabolic subgroup, we may identify the standard tractor bundle $\mathcal{T}$ with the associated bundle $E \times_{\dot{P}} \mathbb{R}^{p+1,q+1}$ induced by the restriction of the standard representation of $\mathrm{O}(p+1, q+1)$ to $\dot{P}$ and $\mathcal{T}^1$ with $\mathcal{D}[-1]$. Likewise, we may identify the adjoint tractor bundle with the associated bundle $E \times_{\dot{P}} \mathfrak{o}(p+1, q+1)$ induced by the adjoint representation.

1.3.2 Splitting of the tractor bundle induced by a conformal representative

Just as a choice of g trivializes $\mathcal{G}$, it induces a splitting of $\mathcal{T}$; this splitting makes convenient some computations, and is furthermore used in the proof of Theorem 2.1.2. The splitting is equivalently given by an isomorphism $\mathcal{T} \cong \mathrm{gr}(\mathcal{T})$, or, via the identifications in the previous subsection and passing to section maps,

$$\mathcal{E}^A \cong \mathcal{E}[1] \oplus \mathcal{E}^a[-1] \oplus \mathcal{E}[-1].$$

Trivializing the weighted bundles using g then gives an (again, equivalent) isomorphism $\mathcal{E}^A \cong \mathcal{E} \oplus \mathcal{E}^a \oplus \mathcal{E}$.

For a given conformal class c, fix a representative metric g. Then, to a tractor $\chi \in \mathcal{E}^A$, which we regard as a map $\mathcal{G} \to T\widetilde{M}|_{\mathcal{G}}$, associate the map $\chi \circ g$; by definition, $(\chi \circ g)(x) \in$

$T_{(x,g_x)}\widetilde{M}$. We can compose with a diffeomorphism of $\widetilde{M}$ fixing $\mathcal{G}$ pointwise to assume that $\widetilde{g}$ is in normal form with respect to g. Then, the splitting $\mathcal{G}\times\mathbb{R}\leftrightarrow\mathbb{R}^+\times M\times\mathbb{R}$ induces a splitting $T(\mathcal{G}\times\mathbb{R})\leftrightarrow T\mathbb{R}^+\oplus TM\oplus T\mathbb{R}$. The coordinate vector fields ∂_t and ∂_ρ trivialize the factors $T\mathbb{R}^+$ and $T\mathbb{R}$, respectively, so, via this trivialization, the map $\chi\circ g$ decomposes as a formal vector function

$$\begin{pmatrix}\chi^0\\ \chi^a\\ \chi^\infty\end{pmatrix}\in\begin{pmatrix}\mathcal{E}\\ \mathcal{E}^a\\ \mathcal{E}\end{pmatrix},$$

Here, χ^0 and χ^∞ respectively denote the $T\mathbb{R}^+$ and $T\mathbb{R}$ factors, so that in local coordinates (x^a) on M, $\chi\circ g$ has representation $\chi^0\partial_0+\chi^a\partial_a+\chi^\infty\partial_\infty$; alternately, denoting $\partial_0:=\partial_t$ and $\partial_\infty:=\partial_\rho$, the representation of $\chi\circ g$ is just $\chi^A\partial_A$. Restricting the normal form (1.18) of $\widetilde{g}$ to $\mathcal{G}=\{\rho=0\}$ gives that in this trivialization, the tractor metric is given by

$$g^{\mathcal{T}}(\alpha,\beta)=\alpha^0\beta^\infty+g_{ab}\alpha^a\beta^b+\alpha^\infty\beta^0.$$

The trivialization $(\chi^0,\chi^a,\chi^\infty)^T$ of $\chi\circ g$ associated to χ depends on the choice of representative $g\in c$; we compute explicitly this dependence.

Proposition 1.3.3. *Given a conformal structure c and representatives g and $\widehat{g}=e^{2\Upsilon}g$, the representations $(\chi^0,\chi^a,\chi^\infty)^T$ and $(\widehat{\chi}^0,\widehat{\chi}^b,\widehat{\chi}^\infty)^T$ in $\mathcal{E}^A\cong\mathcal{E}\oplus\mathcal{E}^a\oplus\mathcal{E}$ of a tractor χ with respect to g and $\widehat{g}$, respectively, are related by*

$$\begin{pmatrix}\widehat{\chi}^0\\ \widehat{\chi}^b\\ \widehat{\chi}^\infty\end{pmatrix}=\begin{pmatrix}e^{-\Upsilon}&0&0\\ 0&e^{-\Upsilon}&0\\ 0&0&e^{\Upsilon}\end{pmatrix}\begin{pmatrix}1&-\Upsilon_a&-\frac{1}{2}\Upsilon_c\Upsilon^c\\ 0&\delta^b{}_a&\Upsilon^b\\ 0&0&1\end{pmatrix}\begin{pmatrix}\chi^0\\ \chi^a\\ \chi^\infty\end{pmatrix}. \tag{1.28}$$

Proof. This proof is adapted from that of [FG, Proposition 6.5], which describes how components of the derivatives of ambient curvature vary with the choice of conformal representative. Let $\widetilde{g}$ be an ambient metric in normal form relative to g, so that $\widetilde{g}$ has the form (1.18). Then, there is a unique homogeneous diffeomorphism ϕ of $\widetilde{M}$ fixing $\mathcal{G}$ pointwise, and with respect to which $\phi^*\widetilde{g}$ is in normal form relative to $\widehat{g}$, so that $\widetilde{g}$ also has the form (1.18) (now in coordinates $(\widehat{t},\widehat{x},\widehat{\rho})$. The two trivializations satisfy $\widehat{t}=e^{-\Upsilon}t$, so there is a homogeneous diffeomorphism $\psi(\widehat{t},\widehat{x},\widehat{\rho})=(t,x,\rho)$ so that $\widetilde{\widetilde{g}}:=\psi^*\widetilde{g}$ also has form (1.18) (in $(\widehat{t},\widehat{x},\widehat{\rho})$) and $\psi(\widehat{t},\widehat{x},0)=(\widehat{t}e^{\Upsilon},\widehat{x},0)$.

Applying $\partial_{\widehat{t}}$ to both sides of the equation gives $(T\psi)^B{}_0 = (e^\Upsilon, 0, 0)^T$, and instead applying $\partial_{\widehat{x}^a} \in \Gamma(TM) \subset \Gamma(T\widetilde{M}|_{\mathcal{G}})$ to both sides gives $T\psi^B{}_a = (e^\Upsilon \widehat{t}\Upsilon_a, \delta^b{}_a, 0)^T$. To compute $(T\psi)^B{}_\infty$, we use that $\psi^*\widetilde{g}|_{\mathcal{G}}$ must have the form

$$\psi^*\widetilde{g}|_{\mathcal{G}} = 2\widehat{t}\,d\widehat{t}\,d\widehat{\rho} + \widehat{t}^2\widehat{g}_{ab}d\widehat{x}^a d\widehat{x}^b.$$

Then, we must have

$$\begin{aligned}\widehat{t} &= (\psi^*\widetilde{g})(\widehat{\partial}_t, \widehat{\partial}_\rho)\\ &= \widetilde{g}(T\psi \cdot \widehat{\partial}_t, T\psi \cdot \widehat{\partial}_\rho)\\ &= e^\Upsilon \widetilde{g}(\partial_t, T\psi \cdot \widehat{\partial}_\rho).\end{aligned}$$

Similarly, for $X \in \Gamma(TM) \subset \Gamma(T\widetilde{M}|_{\mathcal{G}})$,

$$\begin{aligned}0 &= (\psi^*\widetilde{g})(X, \widehat{\partial}_\rho)\\ &= \widetilde{g}(T\psi \cdot X, T\psi \cdot \widehat{\partial}_\rho)\\ &= \widetilde{g}(\widehat{t}e^\Upsilon d\Upsilon(X)\partial_t + X, T\psi \cdot \widehat{\partial}_\rho)\\ &= e^\Upsilon \widehat{t}\,d\Upsilon(X)\widetilde{g}(\partial_t, T\psi \cdot \widehat{\partial}_\rho) + \widetilde{g}(X, T\psi \cdot \widehat{\partial}_\rho).\end{aligned}$$

Similarly, and finally,

$$\begin{aligned}0 &= (\psi^*\widetilde{g})(\widehat{\partial}_\rho, \widehat{\partial}_\rho)\\ &= \widetilde{g}(T\psi \cdot \widehat{\partial}_\rho, T\psi \cdot \widehat{\partial}_\rho).\end{aligned}$$

Solving these equations in succession give that the Jacobian of ψ along $\mathcal{G}$ is

$$(T\psi|_{\mathcal{G}})^B{}_A = \begin{pmatrix} e^\Upsilon & \widehat{t}e^\Upsilon\Upsilon_a & -\frac{1}{2}\widehat{t}e^{-\Upsilon}\Upsilon_c\Upsilon^c \\ 0 & \delta^b{}_a & -e^{-2\Upsilon}\Upsilon^b \\ 0 & 0 & e^{-2\Upsilon} \end{pmatrix}. \tag{1.29}$$

By definition, a tractor χ has homogeneity -1; furthermore, since dt has homogeneity 1 and dx^i and $d\rho$ have homogeneity 0, the component $\widetilde{\chi}^B$ has homogeneity $-1 + s_0$, where $s_0 = 1$ if $B = 0$ and $s_0 = 0$ otherwise, and so

$$\widehat{\chi}^B|_{\{\widehat{t}=1\}} = e^{(-1+s_0)\Upsilon}\widehat{\chi}^B|_{\{\widehat{t}=e^{-\Upsilon}\}}. \tag{1.30}$$

Evaluating both sides of

$$\chi^B \circ \psi = (T\psi)^B{}_A \widehat{\chi}^A$$

at $\widehat{t} = e^{-\Upsilon}$, $\rho = 0$, and substituting using (1.30) gives

$$\left.\begin{pmatrix} \chi^0 \\ \chi^b \\ \chi^\infty \end{pmatrix}\right|_{t=1,\rho=0} = \begin{pmatrix} e^\Upsilon & \Upsilon_a & -\frac{1}{2}e^{-2\Upsilon}\Upsilon^c\Upsilon_c \\ 0 & \delta^b{}_a & -e^{-2\Upsilon}\Upsilon^b \\ 0 & 0 & e^{-2\Upsilon} \end{pmatrix} \left.\begin{pmatrix} \widehat{\chi}^0 \\ e^\Upsilon \widehat{\chi}^b \\ e^\Upsilon \widehat{\chi}^\infty \end{pmatrix}\right|_{\widehat{t}=1,\rho=0}$$

Multiplying and solving for $\widehat{\chi}^B|_{\widehat{t}=1,\rho=0}$ gives the desired identity. □

Reading off the homogeneities in (1.28) shows that we may naturally regard the components of χ in the splitting as sections of weighted bundles, namely, as $\chi^0 \in \mathcal{E}[-1]$, $\chi^a \in \mathcal{E}^a[-1]$, and $\chi^\infty \in \mathcal{E}[1]$, and the unweighted versions of the components are merely $\chi^0 \circ g$, $\chi^a \circ g$, and $\chi^\infty \circ g$. So, a choice of representative $g \in c$ induces an isomorphism $\mathcal{E}^A \cong \mathcal{E}[1] \oplus \mathcal{E}^a[-1] \oplus \mathcal{E}[-1]$, that is, an isomorphism $\mathcal{T} \cong \operatorname{gr}(\mathcal{T})$.

The transformation law for the weighted versions of the components is

$$\begin{pmatrix} \widehat{\chi}^0 \\ \widehat{\chi}^b \\ \widehat{\chi}^\infty \end{pmatrix} = \begin{pmatrix} 1 & -\Upsilon_a & -\frac{1}{2}\Upsilon_c\Upsilon^c \\ 0 & \delta^b{}_a & \Upsilon^b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \chi^0 \\ \chi^a \\ \chi^\infty \end{pmatrix}.$$

In this splitting, the canonical projection $\Pi_0 : \mathcal{E}^A \to \mathcal{E}[1]$ is just $\chi \mapsto \chi^\infty$, and the above transformation law illustrates that $\chi^\infty \in \mathcal{E}[1]$ is independent of the choice of representative. We thus call χ^∞ the **primary part** of χ (whether we regard χ^∞ as a weighted or unweighted section). If $\chi^\infty = 0$, then the transformation law shows that $\chi^a \in \mathcal{E}^a[-1]$ is independent of the choice of representative $g \in c$, and if $\chi^\infty = 0$ and $\chi^a = 0$, then $\chi^0 \in \mathcal{E}[-1]$ is independent of g. In the splitting, the canonical injection $\iota : \mathcal{E}[-1] \to \mathcal{E}^A$ is just the map

$$\iota : \sigma \mapsto \begin{pmatrix} \sigma \\ 0 \\ 0 \end{pmatrix}.$$

We now compute the tractor connection $\nabla^{\mathcal{T}}$ in terms of the above splitting. By construction, to describe the splitting of the covariant derivative of a tractor (into unweighted components), we just evaluate $\widetilde{\Gamma}^C_{AB}$ at $t = 1$, $\rho = 0$, and (because the associated tractor covariant derivative is just defined for directions tangent to M) keep only the $A = a$ components. Consulting (1.22) and substituting $g'_{ab} = 2P_{ab}$ gives

$$\begin{aligned}\widetilde{\Gamma}^0_{aB} &= \begin{pmatrix} 0 & -P_{ab} & 0 \end{pmatrix}\\ \widetilde{\Gamma}^c_{aB} &= \begin{pmatrix} \delta_a{}^c & \Gamma^c_{ab} & P_a{}^c \end{pmatrix}\\ \widetilde{\Gamma}^0_{aB} &= \begin{pmatrix} 0 & -g_{ab} & 0 \end{pmatrix}.\end{aligned}$$

Substituting gives that, in terms of the splitting induced by g, the tractor covariant derivative is

$$\begin{pmatrix} \chi^0 \\ \chi^a \\ \chi^\infty \end{pmatrix}_{,b} = \begin{pmatrix} (\chi^0)_b - P_{bc}\chi^c \\ \chi^a{}_{,b} + \delta_b{}^a\chi^0 + P_b{}^a\chi^\infty \\ (\chi^\infty)_b - \chi_b \end{pmatrix}. \tag{1.31}$$

The comma on the left-hand side denotes the tractor covariant derivative, and the commas in the arguments on the right-hand side denote the Levi-Civita covariant derivative of g. By construction, if we interpret that latter derivative appropriately, we can interpret all the components of χ appearing in (1.31) as weighted sections.

Computing directly using this formula gives the components of the tractor curvature in the induced splitting:

$$\begin{aligned} R^{\mathcal{T}}_{ab0D} &= 0\\ R^{\mathcal{T}}_{abcd} &= W_{abcd}\\ R^{\mathcal{T}}_{ab\infty d} &= C_{dab},\end{aligned}$$

where W and C are respectively the Weyl and Cotton tensors of the representative $g \in c$ (cf. Example 1.1.21). Alternately, because $\nabla^{\mathcal{T}}$ is essentially given by restricting $\widetilde{\nabla}$, by construction we may simply read these formulas directly from (1.25). By construction, W has conformal weight 2.

Recall from the beginning of the section that we call the dual $\mathcal{T}^*$ of the tractor bundle the cotractor bundle, and that we call its sections cotractors or tractor 1-forms. For convenience, we prove results in this work mostly in terms of cotractors, so we collect here dualizations of some of the constructions already described for tractors.

A choice of representative $g \in c$ induces a splitting $\mathcal{E}_A \leftrightarrow \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$, and we denote the components of the splitting using formal row vectors,

$$\chi_A \leftrightarrow \begin{pmatrix} \chi_0 & \chi_a & \chi_\infty \end{pmatrix}.$$

As above, we can trivialize the weight bundles using g again. Then, using our conventions, the natural pairing of a cotractor α_A and a tractor β^B is just given by formal matrix multiplication:

$$\alpha_A \beta^A = \alpha_0 \beta^0 + \alpha_a \beta^a + \alpha_\infty \beta^\infty = \begin{pmatrix} \alpha_0 & \alpha_a & \alpha_\infty \end{pmatrix} \begin{pmatrix} \beta^0 \\ \beta^a \\ \beta^\infty \end{pmatrix}.$$

In terms of such a splitting, the covariant derivative is given by

$$\begin{pmatrix} \chi_0 & \chi_a & \chi_\infty \end{pmatrix}_{,b} = \begin{pmatrix} (\chi_0)_b - \chi_b & \chi_{a,b} + g_{ab}\chi_0 + P_{ab}\chi_\infty & (\chi_\infty)_b - P_b{}^c \chi_c \end{pmatrix}. \tag{1.32}$$

The weighted component $\chi_0 \in \mathcal{E}[1]$ of χ is independent of the choice of representative $g \in c$, and we call χ_0 the **primary part** of χ. The corresponding **canonical projection** is the map $\Pi_0 : \mathcal{E}_A \to \mathcal{E}[1]$ given by $\Pi_0 : \chi_A \mapsto \chi_A \mathbb{T}^A$, and the **canonical injection** $\iota : \mathcal{E}[-1] \to \mathcal{E}_A$ is given by $f \mapsto f\mathbb{T}_A$.

1.3.3 Conformally Einstein metrics

In this section we give a brief account of the standard realization of the tractor bundle, mostly following [Gov], highlighting the role of Einstein metrics in tractor geometry, and showing that it agrees with the construction given in the previous subsection.

Recall that a metric is Einstein iff $P_{ab} = \lambda g_{ab}$ for some constant λ, and that a conformal class is Einstein if it contains an Einstein representative. Taking traces shows that a metric g is Einstein iff

$$P_{ab} - \frac{1}{n} P_d{}^d g_{ab} = 0. \tag{1.33}$$

So, to determine whether c is Einstein using an arbitrary representative $g \in c$, we analyze how P changes with the change of representative from g to $\widehat{g}$ and then check whether there is some representative $\widehat{g} \in c$ such that the Schouten tensor $\widehat{P}$ of $\widehat{g}$ satisfies (1.33).

For an arbitrary representative $g \in c$, we may write $\widehat{g} = e^{2\Upsilon} g$ for some function $\Upsilon \in \mathcal{E}$ (and conversely any such choice of Υ gives a conformal representative). Computing gives

$$\widehat{P}_{ab} = P_{ab} - \Upsilon_{ab} + \Upsilon_a \Upsilon_b. \tag{1.34}$$

Substituting this equation in (1.33) yields an overdetermined partial differential equation in Υ, and by construction c is Einstein iff it admits a global solution. This equation is linear with respect to $\sigma := e^{-\Upsilon}$ (so, that the corresponding representative is $\widehat{g} = \sigma^{-2} g$); it becomes

$$(\sigma_{ab} + P_{ab}\sigma) - \frac{1}{n}(\sigma_d{}^d + P_d{}^d\sigma) g_{ab} = 0, \tag{1.35}$$

so that the left-hand side is just the g-tracefree part of $\sigma_{ab} + P_{ab}\sigma$. By construction, this condition is conformally invariant: If σ is a solution to the equation (1.35) for a choice of representative g, then $\Omega\sigma$ is a solution to the equation for the representative $\Omega^2 g$. We may frame this more invariantly using weighted bundles: Recall that we may identify c with the canonical bilinear form $\mathbf{g} \in \mathcal{E}_{(ab)}[2]$. The conformal class c is Einstein iff there is a (nonvanishing) section $\sigma \in \mathcal{E}[1]$ such that the metric $\sigma^{-2}\mathbf{g}$ is Einstein; we call such a section an **Einstein scale**.

Even though (1.35) is overdetermined and does not generally admit (even local) solutions, it defines fiberwise a conformally invariant subbundle of the 2-jet bundle $J^2\mathcal{D}[1]$.

Definition 1.3.4. The **(standard) tractor bundle** $\mathcal{T} \to M$ of a conformal structure (M, c) is the subbundle of the bundle $J^2\mathcal{D}[1]$ of 2-jets satisfying (1.35). A **(standard) tractor** is a section of $\mathcal{T}$.

We show that this definition of the tractor bundle agrees with the one in subsection 1.3.1.

Introducing the auxiliary 1-form $\mu_a := \sigma_a \in \mathcal{E}_a[1]$ realizes (1.35) as a first-order system,

$$\begin{aligned} \sigma_b - \mu_b &= 0 \\ \mu_{a,b} + P_{ab}\sigma + \rho \mathbf{g}_{ab} &= 0, \end{aligned} \tag{1.36}$$

where $\rho := -\frac{1}{n}(\sigma_c{}^c + P_c{}^c\sigma) \in \mathcal{E}[-1]$. Alternately, we may view these equations as pointwise conditions defining the subbundle $\mathcal{T} \subset J^2\mathcal{D}[1]$ and coordinates $(\sigma, \mu_a, \rho)^T$ on that subbundle.

Raising the index of μ produces a section $\mu^a \in \mathcal{E}[-1]$; then, computing directly shows that triples $(\sigma, \mu^a, \rho)^T$ satisfying those equations transform under a change of representative $g \in c$ precisely as triples $(\chi^0, \chi^a, \chi^\infty)^T$ do; since this transformation characterizes the previous construction of the tractor bundle, the two definitions of $\mathcal{T}$ agree.

Using this latter formulation of the tractor bundle, we can give an illustrative construction of the tractor connection, $\nabla^{\mathcal{T}}$.

We now manipulate the system to produce a linear equation in ρ_b but no new auxiliary variables, which will thus yield a closed, first-order linear system in (σ, μ_a, ρ). First, substituting the first equation in the second gives an equation just in σ and ρ:

$$\sigma_{ab} + P_{ab}\sigma + \rho g_{ab} = 0. \tag{1.37}$$

Differentiating (using c for the derivative index), and applying the Ricci identity to the resulting term σ_{abc} gives

$$(\sigma_{acb} + R^d{}_{abc}\sigma_d) + P_{ab}\sigma_c + P_{ab,c}\sigma + g_{ab}\rho_c = 0.$$

Contracting with g^{ac} and using the characterization (1.13) of the Schouten tensor gives

$$\begin{aligned} 0 &= \sigma^c{}_{cb} + R^d{}_b\sigma_d + P^c_b\sigma_c + P^c_{b,c}\sigma + \rho_b \\ &= \sigma^c{}_{cb} + (n-2)P^c_b\sigma_c + P^c_c\sigma_b + P^c_b\sigma_c + P^c_{b,c}\sigma + \rho_b. \end{aligned}$$

Finally, contracting (1.37) with g^{ab} and solving for the Laplacian of σ gives

$$\sigma^c{}_c = -P^c_c\sigma - n\rho.$$

Substituting this identity into the previous display equation, applying the product rule, and solving for ρ_b gives

$$\rho_b - P_{bc}\mu^c = 0.$$

Adding this equation to the previous system then gives the system:

$$\begin{aligned} \sigma_b - \mu_b &= 0 \\ \mu_{a,b} + P_{ab}\sigma + \rho\mathbf{g}_{ab} &= 0 \\ \rho_b - P_{bc}\mu^c &= 0. \end{aligned} \tag{1.38}$$

So, the Einstein condition defines a map

$$\nabla^{\mathcal{T}} : \Gamma(\mathcal{D}[1] \otimes T^*M[1] \otimes \mathcal{D}[-1]) \to \Gamma((\mathcal{D}[1] \otimes T^*M[1] \otimes \mathcal{D}[-1]) \otimes T^*M)$$

which maps the triple (σ, μ_a, ρ) to the formal vector whose entries are left-hand sides of the equations in the system (1.38). This is just (1.32), and so this definition of $\nabla^{\mathcal{T}}$ agrees with the one in Subsection 1.3.1.

So, given a parallel tractor $\chi \in \Gamma(\mathcal{T})$, its triple $(\sigma, \mu_a, \rho)^T$ of components with respect to the splitting induced by an arbitrary choice $g \in c$ satisfies the system. Thus, if also $\Pi_0(\chi)$ is nonvanishing, it is an Einstein scale. Conversely, if $\sigma \in \mathcal{E}[1]$ is an Einstein scale, then by the construction in this subsection, for an arbitrary choice of representative $g \in c$, there are sections $\mu_a \in \mathcal{E}_a[1]$ and $\rho \in \mathcal{E}[-1]$ satisfying the closed, first-order system (1.38), or equivalently, such that the cotractor whose decomposition with respect to g is (σ, μ_a, ρ) is parallel. Rearranging the first two equations in (1.38) (and taking the trace of the second equation) we can recover μ^a and ρ:

$$\begin{aligned} \mu^a &= \sigma^a \\ \rho &= -\tfrac{1}{n}(\sigma_b{}^b + P_b{}^b \sigma). \end{aligned}$$

Of course, these definitions make sense for all σ, not just Einstein scales.

Definition 1.3.5. The **(first) BGG splitting operator** (for cotractors) is the map $L_0 : \mathcal{E}[1] \to \mathcal{E}_A$ defined (with respect to the splitting induced by an arbitrary representative $g \in c$) by

$$L_0(\sigma)_A := \begin{pmatrix} \sigma & \sigma_a & -\tfrac{1}{n}(\sigma_b{}^b + P_b{}^b \sigma) \end{pmatrix}. \tag{1.39}$$

So, by construction, if χ is parallel, $L_0 \circ \Pi_0 = \mathrm{id}_{\mathcal{T}}$. Summarizing gives the following:

Proposition 1.3.6. *Let c be a conformal structure. The restrictions of the maps Π_0 and L_0 define a bijective correspondence between Einstein scales (and thus Einstein representatives of c) and $\nabla^{\mathcal{T}}$-parallel sections of $\mathcal{T}$ whose primary part is nonvanishing.*

If $g \in c$ is Einstein, then $\xi = 1$ satisfies (1.35) (if regarded as a weighted section, ξ is the unique section such that $\xi^{-2}\mathbf{g} = g$). With respect to the decomposition induced by the choice $g \in c$, the corresponding parallel cotractor is $L_0(\xi) = (1, 0, -\lambda)$, or $L_0(\xi) = dt - t\lambda d\rho$. Computing gives $L_0(\xi)^A L_0(\xi)_A = -2\lambda$.

The mild awkwardness of the restriction in the previous proposition to nonvanishing sections suggests a natural generalization of Einstein scales.

Definition 1.3.7. An **almost Einstein scale** is a (weighted or unweighted) solution of (1.35) that is not identically zero.

Expanding attention to include almost Einstein structures yields the following analogue of Proposition 1.3.6, which follows from the same arguments.

Proposition 1.3.8. *Let c be a conformal structure. The restrictions of the maps Π_0 and L_0 define a bijective correspondence between almost Einstein scales and nonzero $\nabla^{\mathcal{T}}$-parallel sections of $\mathcal{T}$.*

The **singular set** of an almost Einstein scale is just its zero set.

Proposition 1.3.9. *The complement of the singular set of an almost Einstein scale is dense.*

Proof. Suppose the singular set of the almost Einstein scale σ is not dense. Then, there is some nonempty open subset on which σ is identically zero. At any point x of that subset, the 2-jet of σ is zero, and so $L_0(\sigma)_x = 0$. Since $L_0(\sigma)$ is parallel, it is identically zero, and thus so is σ, contradicting that σ is an almost Einstein scale. □

Given a conformal structure c, an almost Einstein scale σ, say, with singular set S, defines an Einstein metric $g := (\sigma|_{M-S})^{-2}(\mathbf{g}|_{M-S})$ on the dense subset $M - S \subseteq M$, but this metric cannot be extended to any larger set of M. By the computation after Proposition 1.3.6, if g has Einstein constant λ, then the function $-\frac{1}{2}L_0(\sigma)^A L_0(\sigma)_A$ extends that constant function to all of M without reference to the singular set [Gov]. We again refer to λ as the **Einstein constant** of σ.

We may frame the conformally almost Einstein condition (1.35) more invariantly. Proposition 1.3.8 shows that a nonzero weighted function $\sigma \in \mathcal{E}[-1]$ is an almost Einstein scale iff $(\nabla^{\mathcal{T}} \circ L_0)(\sigma) = 0$. Computing in a splitting induced by a choice $g \in c$ gives

$$[L_0(\sigma)]_{A,b} = \begin{pmatrix} 0 \\ (\sigma_{ab} + P_{ab}\sigma) - \frac{1}{n}(\sigma_c{}^c + P_c{}^c\sigma)\mathbf{g}_{ab} \\ -\frac{1}{n}(\sigma_c{}^c + P_c{}^c\sigma)_b - P_b{}^c\sigma_c \end{pmatrix}^T$$

The condition that the middle component vanish is precisely the almost Einstein condition, (1.35). The vanishing of the bottom component is a differential consequence of the vanishing of the middle component, recovering in tractor language the fact that that almost Einstein scales correspond to parallel tractors. Since the primary part of $[L_0(\sigma)]_{A,b}$ is zero, by construction the projection Π_1 onto the middle component (which takes values in $\mathcal{E}_{(ab)_0} = \Gamma(S^2_0 T^*M))$ is independent of choice of representative $g \in c$, recovering the fact that the almost Einstein condition is conformally invariant. Note that we may write the almost Einstein condition invariantly as $\Theta_0(\sigma) = 0$, where

$$\Theta_0 := \Pi_1 \circ \nabla^{\mathcal{T}} \circ L_0$$

is the **(first) BGG operator** (for cotractors):

$$\begin{array}{ccc} \mathcal{E}_A & \xrightarrow{\nabla^{\mathcal{T}}} & \mathcal{E}_{Ab} \\ {\scriptstyle L_0}\uparrow & & \downarrow{\scriptstyle \Pi_1}. \\ \mathcal{E}[1] & \xrightarrow[\Theta_0]{} & \mathcal{E}_{(ab)_0} \end{array}$$

Here, $\mathcal{E}_{Ab} = \Gamma(\mathcal{T} \otimes TM)$.

1.3.4 Higher-rank tractor tensors

We can extend many of the results in the previous subsections to tractor tensors of more types, and in some cases to those induced by arbitrary representations. We state results here just for covariant tractor tensor bundles, and can recover the corresponding results for contravariant and mixed tractor tensor bundles just by raising indices.

The decomposition $\mathcal{E}_A \cong \mathcal{E}[1] \oplus \mathcal{E}_a[1] \oplus \mathcal{E}[-1]$ induced by a choice of representative $g \in c$ yields decompositions of arbitrary tractor tensor bundles. For example, we may denote for

$r > 1$ a tractor r-form $\chi_{A_1\cdots A_r}$, that is, a section in $\mathcal{E}_{[A_1\cdots A_r]} = \Gamma(\Lambda^r \mathcal{T}^*)$, by

$$\begin{pmatrix} & \chi_{a_1\cdots a_r} & \\ \chi_{0a_2\cdots a_r} & & \chi_{\infty a_2\cdots a_r} \\ & \chi_{0\infty a_3\cdots a_r} & \end{pmatrix}.$$

Recall that we denote $\mathcal{E}_{[b_1\cdots b_k]} := \Gamma(\Lambda^k T^* M)$. Consulting the homogeneities of the bundles in the decomposition of the cotractor bundle gives that $\mathcal{E}_{[A_1\cdots A_r]}$ decomposes as

$$\mathcal{E}_{[A_1\cdots A_r]} = \begin{pmatrix} & \mathcal{E}_{[a_1\cdots a_r]}[r] & \\ \mathcal{E}_{[a_2\cdots a_r]}[r] & & \mathcal{E}_{[a_2\cdots a_r]}[r-2] \\ & \mathcal{E}_{[a_3\cdots a_r]}[r-2] & \end{pmatrix}.$$

Composing the components with g to produce unweighted forms shows that the expression for

$$\chi = \begin{pmatrix} & \chi_{a_1\cdots a_r} & \\ \chi_{0a_2\cdots a_r} & & \chi_{\infty a_2\cdots a_r} \\ & \chi_{0\infty a_3\cdots a_r} & \end{pmatrix}$$

in a local coframe of the form $(dt, dx^a, d\rho)$ is just

$$t^{r-1}\chi_{0a_2\cdots a_r} dt \wedge dx^{a_2} \wedge \cdots \wedge dx^{a_r} + t^r \chi_{a_1\cdots a_r} dx^{a_1} \wedge \cdots \wedge dx^{a_r}$$
$$+ t^{r-1}\chi_{0\infty a_3\cdots a_r} dt \wedge d\rho \wedge dx^{a_3} \cdots \wedge dx^{a_r} + t^r \chi_{\infty a_2\cdots a_r} d\rho \wedge dx^{a_2} \wedge \cdots \wedge dx^{a_r}.$$

Now, let $\mathbb{W}$ be any representation of $\mathrm{O}(p+1, q+1)$ (recall that in this dissertation, all representations are finite-dimensional). Denote the nilpotent part of $\dot{P}$ by $\dot{N}$ and its nilpotent part by $\dot{\mathfrak{n}}$. Since $\dot{P}$ normalizes $\dot{M}$, $\dot{\mathfrak{n}}.\mathbb{W}$ is a $\dot{P}$-subrepresentation of $\mathbb{W}$, yielding a natural $\dot{P}$-equivariant quotient map

$$\mathbb{W} \to \mathbb{W}_{\dot{\mathfrak{n}}} := \mathbb{W}/(\dot{\mathfrak{n}}.\mathbb{W}).$$

Now, for any conformal structure c, let $(E \to M, \omega)$ be the corresponding normal $(\mathfrak{o}(p+1, q+1), \dot{P})$-geometry. For the associated bundles $W := E \times_{\dot{P}} \mathbb{W}$ and $W_{\dot{\mathfrak{n}}} := E \times_{\dot{P}} \mathbb{W}_{\dot{\mathfrak{n}}}$, the quotient map $\mathbb{W} \to \mathbb{W}_{\dot{\mathfrak{n}}}$ induces a map

$$\Pi_0 : W \to W_{\dot{\mathfrak{n}}}.$$

This map and its section map, also denoted Π_0, are called the **canonical projection** of W. We call the image of a section $\chi \in \Gamma(W)$ under Π_0 the **primary part** of χ. For convenience we henceforth implicitly dualize with $g^{\mathcal{T}}$ as needed and regard the induced bundles W as subbundles of tensor powers $\bigotimes^r \mathcal{T}^*$ of the cotractor bundle $\mathcal{T}^*$.

Example 1.3.10. If $\mathbb{W} = \bullet$ (that is, the trivial representation), then $\Gamma(W) = \mathcal{E}$ and $\dot{\mathfrak{n}}$ trivially annihilates $\mathbb{W}$, so $\dot{\mathfrak{n}}.\mathbb{W} = 0$ and thus Π_0 may be identified with the identity map.

Example 1.3.11. If $\mathbb{W} = \square$ (the dual of the standard representation), then $\Gamma(W) = \mathcal{E}_A$. A straightforward computation shows that $\Gamma(W_{\dot{\mathfrak{n}}}) = \mathcal{E}[1]$, and the canonical projection Π_0 coincides with the map so named in Subsection 1.3.1.

Example 1.3.12. If $\mathbb{W} = \left.\begin{matrix}\square \\ \vdots \\ \square\end{matrix}\right\} r$ (the antisymmetric representation $\Lambda^r(\mathbb{R}^{p+1,q+1})^*)$, then $\Gamma(W) = \mathcal{E}_{[A_1 \cdots A_r]}$. A short algebraic argument shows that we may identify $\Gamma(W_{\dot{\mathfrak{n}}}) = \mathcal{E}_{[a_2 \cdots a_r]}[r]$ and that the canonical projection is

$$\Pi_0 : \chi_{A_1 \cdots \mapsto A_r} \mapsto \chi_{0a_2 \cdots a_r}.$$

Notice that we may also write this map as $\chi \mapsto \iota^*(\mathbb{T} \lrcorner \chi)$, where ι is the inclusion $\mathcal{G} \to \widetilde{M}$. By construction, the right-hand side has the indicated weight. (Note that setting $r = 1$ just recovers the previous example.)

The tractor connection $\nabla^{\mathcal{T}}$ induces connections (all also denoted $\nabla^{\mathcal{T}}$) on tractor tensor bundles W induced by $\mathrm{O}(p+1, q+1)$-representations $\mathbb{W}$. They are characterized by the identities

$$\nabla^{\mathcal{T}} f = df$$

for $f \in \mathcal{E}$ and

$$\nabla^{\mathcal{T}}(\alpha \otimes \beta) = \alpha \otimes \nabla^{\mathcal{T}} \beta + \nabla^{\mathcal{T}} \alpha \otimes \beta$$

for arbitrary tractor tensors α and β.

Example 1.3.13. Computing directly shows that the connections on the tractor r-form bundles $\Lambda^r\mathcal{T}^*$, $r > 1$ (which again are induced by the representations $\Lambda^r\mathbb{V}$) are given by

$$\left[\begin{pmatrix} \chi_{0a_2\cdots a_r} \\ \chi_{a_1\cdots a_r} \,\big|\, \chi_{0\infty a_3\cdots a_r} \\ \chi_{\infty a_2\cdots a_r} \end{pmatrix}^T\right]_{,b} = \begin{pmatrix} \chi_{0a_2\cdots a_r,b} - \chi_{a_2\cdots a_r b} + (r-1)\mathbf{g}_{b[a_2}\chi_{|0\infty|a_3\cdots a_r]} \\ \begin{pmatrix} \chi_{a_1\cdots a_r,b} + r\mathbf{g}_{b[a_1}\chi_{|\infty|a_2\cdots a_r]} \\ +rP_{b[a_1}\chi_{|0|a_2\cdots a_r]} \end{pmatrix} \Bigg| \begin{pmatrix} \chi_{0\infty a_3\cdots a_r,b} + \chi_{\infty b a_3\cdots a_r} \\ -P_b{}^c\chi_{0ca_3\cdots a_r} \end{pmatrix} \\ \chi_{\infty a_2\cdots a_r,b} - P_b{}^c\chi_{ca_2\cdots a_r} - (r-1)P_{b[a_2}\chi_{|0\infty|a_3\cdots a_r]} \end{pmatrix}^T . \quad (1.40)$$

The comma on the left-hand side denotes the tractor covariant derivative. On the right-hand side, the commas denote the Levi-Civita connections on the respective bundles $\Lambda^q T^*M$, and the 0 and ∞ indices indicate the component of $\chi_{A_1\cdots A_r}$, but do not function as indices of the component function. So, for example, if $\sigma_{a_2\cdots a_r} = \chi_{0a_2\cdots a_r}$, then $\chi_{0a_2\cdots a_r,b}$ just denotes $\sigma_{a_2\cdots a_r,b}$, not the $0a_2\cdots a_r b$ component of $\nabla^{\mathcal{T}}\chi$.

The BGG splitting operator generalizes to general tractor tensor bundles too; specializing Theorem 2.5 and Lemma 2.7(2) in [ČSS01] to the present setting gives the following:

Theorem 1.3.14. *Suppose $\mathbb{W}$ is an irreducible $\mathrm{O}(p+1,q+1)$-representation. There is a linear natural differential operator $L_0 : W_{\dot{\mathfrak{n}}} \to W$, called the* ***(first) BGG splitting operator****, with the property that if $\chi \in \Gamma(W)$ satisfies $\nabla^{\mathcal{T}}\chi = 0$, then $L_0(\Pi_0(\chi)) = \chi$.*

More concretely, this theorem just says that any parallel section of a tractor tensor bundle induced by an irreducible $\mathrm{O}(p+1,q+1)$-representation can be recovered from its projecting part. Strictly speaking, as stated in [ČSS01] this theorem holds for different parabolic subgroup of $\mathrm{O}(p+1,q+1)$ with the same Lie algebra $\dot{\mathfrak{p}}$, but it appears to hold just as well for $\dot{P}$; we henceforth assume that it does.

Example 1.3.15. If $\mathbb{W} = \bullet$, then because Π_0 is the identity, so is L_0.

Example 1.3.16. If $\mathbb{W} = \square$, then the BGG splitting operator $L_0 : \mathcal{E}[1] \to \mathcal{E}_A$ is just the so-named map in (1.39).

Example 1.3.17. If $\mathbb{W} = \left.\begin{matrix}\square\\ \vdots \\ \square\end{matrix}\right\} r$, then the BGG splitting operator $L_0 : \mathcal{E}_{[a_2\cdots a_r]}[r] \to \mathcal{E}_{[A_1\cdots A_r]}$ can be computed by writing the condition that the parallel tractor tensor be parallel in components with respect to a splitting induced by an arbitrary choice of representative $g \in c$ and then solving for the components of successively higher strengths. Doing so gives that L_0 is given by [HS09]

$$L_0(\sigma)_{A_1\cdots A_r}$$
$$= \begin{pmatrix} \sigma_{a_2\cdots a_r} \\ \sigma_{[a_1\cdots a_{r-1},a_r]} \Big| -\tfrac{1}{n-k+1}\sigma_{ba_3\cdots a_r,}{}^{b} \\ \begin{pmatrix} -\tfrac{1}{r}\sigma_{a_2\cdots a_r,b}{}^{b} + \tfrac{r-1}{nr}\sigma_{p[a_3\cdots a_r|b|,a_2]}{}^{b} + \tfrac{r-1}{n(n-r+2)}\sigma_{[a_3\cdots a_r|b|,\ a_2]}{}^{b} \\ +\tfrac{2(r-1)}{n}P^{b}{}_{[a_2}\sigma_{|b|a_3\cdots a_r]} - \tfrac{1}{n}P_{b}{}^{b}\sigma_{a_2\cdots a_r} \end{pmatrix} \end{pmatrix}^{T}. \quad (1.41)$$

Applying the argument in the proof of Proposition 1.3.9 gives the following result:

Proposition 1.3.18. *Suppose* $\mathbb{W}$ *is an irreducible* $\mathrm{O}(p+1, q+1)$*-representation and let* W *be the induced tractor tensor bundle. If* $\chi \in \Gamma(W)$ *is parallel and nonzero, then the projecting part* $\Pi_0(\chi)$ *of* χ *is nonzero on a dense open set.*

Remark 1.3.19. We may verify this directly for $\mathbb{W} = \left.\begin{matrix}\square\\ \vdots \\ \square\end{matrix}\right\} r$ using (1.41).

Now, for $r > 1$, let Π_1 denote the map $\mathcal{E}_{[A_1\cdots A_r]b} \to \mathcal{E}_{[a_2\cdots a_r]b}[r]$ defined by

$$\Pi_1 : \chi_{A_1\cdots A_r b} \mapsto \chi_{0a_2\cdots a_r b}.$$

By construction (or just checking directly), this projection is independent of the choice of representative. Then, define the **(first) BGG operator** by

$$\Theta_0 := \Pi_1 \circ \nabla^{\mathcal{T}} \circ L_0 : \mathcal{E}_{[a_2\cdots a_r]}[r] \to \mathcal{E}_{[a_2\cdots a_r]b}[r].$$

(This form analogizes but does not generalize the BGG splitting operator defined for cotractors by (1.39).) Computing directly using (1.40) and (1.41) gives

$$\Theta_0(\sigma)_{a_2\cdots a_r b} = \sigma_{a_2\cdots a_r,b} - \sigma_{[a_2\cdots a_r,b]} - \frac{r-1}{n-r+2} g_{b[a_2}\sigma_{|c|a_3\cdots a_r],}{}^{c}. \quad (1.42)$$

We may realize Θ_0 in another way: Take $\mathbb{V} = (\mathbb{R}^{p,q})^*$. Then, the decomposition of the representation $\Lambda^{r-1}\mathbb{V} \otimes \mathbb{V}$, $r > 1$, into irreducible $\mathrm{O}(p,q)$-representations is

$$\Lambda^{r-1}\mathbb{V} \otimes \mathbb{V} \cong \Lambda^r\mathbb{V} \oplus \Lambda^{r-2}\mathbb{V} \oplus \Lambda^{r-1,1}\mathbb{V},$$

where $\Lambda^{r-1,1}\mathbb{V}$ denotes the representation

$$(r-1)\left\{ \begin{array}{|c|c|} \hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \multicolumn{1}{c}{\vdots} & \multicolumn{1}{c}{} \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \right._{0},$$

(which comprises the tensors $\alpha \in \Lambda^{r-1}\mathbb{V} \otimes \mathbb{V}$ satisfying $\alpha_{[a_2\cdots a_r b]} = 0$ and $\alpha_{a_2\cdots a_{r-1}b}{}^{b} = 0$). This decomposition induces [Sem03] a corresponding decomposition

$$\mathcal{E}_{[a_2\cdots a_r]b} \cong \mathcal{E}_{[a_2\cdots a_r b]} \oplus \mathcal{E}_{[a_2\cdots a_{r-1}]} \oplus \mathcal{E}_{\{[a_2\cdots a_r]b\}_0}.$$

Here, $\mathcal{E}_{\{[a_2\cdots a_r]b\}_0}$ denotes the subspace of tractor tensors in $\Lambda^{r-1}\mathcal{T}^* \otimes \mathcal{T}^*$ formally satisfying the same identities as α. Consulting (1.42), $\Theta_0(\sigma)$ is just given by applying the projection $\mathcal{E}_{[a_2\cdots a_r]b}[r] \mapsto \mathcal{E}_{\{[a_2\cdots a_r]b\}_0}[r]$ to $(\nabla^{\mathcal{T}} L_0(\sigma))_{[a_2\cdots a_r]b}$.

$$\begin{array}{ccc} \mathcal{E}_{[A_1\cdots A_r]} & \xrightarrow{\nabla^{\mathcal{T}}} & \mathcal{E}_{[A_1\cdots A_r]b} \\ {\scriptstyle L_0}\big\uparrow & & \big\downarrow{\scriptstyle \Pi_1} \\ \mathcal{E}_{[a_2\cdots a_r]}[r] & \xrightarrow[\Theta_0]{} & \mathcal{E}_{\{[a_2\cdots a_r]b\}_0}[r] \end{array}.$$

Definition 1.3.20. A **conformal Killing** $(r-1)$**-form**, $r > 1$, is a section of in the kernel of $\Theta_0 : \mathcal{E}_{a_2\cdots a_r}[r] \to \mathcal{E}_{\{[a_2\cdots a_r]b\}_0}[r]$. In this context, we call $\Theta_0(\sigma) = 0$ the conformal Killing form equation. A **normal conformal Killing** $(r-1)$**-form** (see [Lei04]), $r > 1$, is a section in $\mathcal{E}_{[a_2\cdots a_r]}[r]$ in the kernel of $\nabla^{\mathcal{T}} \circ L_0 : \mathcal{E}_{a_2\cdots a_r}[r] \to \mathcal{E}_{[A_1\cdots A_r]b}$. So, any normal conformal Killing form is a conformal Killing form.

An argument essentially identical to the one that gave Proposition 2.2.9 yields the following analogue:

Proposition 1.3.21. *Let c be a conformal structure and fix $r > 1$. The restrictions of the maps Π_0 and L_0 define a bijective correspondence between normal conformal Killing $(r-1)$-forms and parallel tractor r-forms.*

1.4 *The groups* G_2 *and* P

The primary application of the main theorem (Subsection 2.1.1) given in this work is the construction of a large family of pseudo-Riemannian metrics with holonomy contained in group (split) G_2, and for which moreover the holonomy is generically equal to that group. The model for the geometric structures exploited to produce these metrics is a natural, G_2-invariant 2-plane field on the homogeneous space G_2/P, where P is a particular 9-dimensional subgroup of P. In this section we describe these objects in further detail.

1.4.1 The group G_2

Complex simple Lie algebras can be classified into four infinite series of "classical" algebras and five exceptional algebras. The complexification of a real simple Lie algebra is either a complex simple Lie algebra or (if the real algebra is already such a complex algebra) a direct sum of two copies of such an algebra, and so real simple Lie algebras may be classified by determining the (always finitely many) real forms of each complex simple Lie algebra, that is, real Lie algebras whose complexification is a given simple complex Lie algebra. The (unique) smallest-dimensional exceptional complex simple Lie algebra, $\mathfrak{g}_2^{\mathbb{C}}$, has two real forms, and the Lie algebra of the group denoted G_2 in this paper is one of those two forms.

Definition 1.4.1. A **composition algebra** over $\mathbb{R}$ is an algebra over $\mathbb{R}$ with a unit and a nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ satisfying

$$\langle xy, xy \rangle = \langle x, x \rangle \langle y, y \rangle$$

for all x, y in the algebra.

The facts here about composition algebras may be found for example in [Har90], where they are called *normed algebras*.

The bilinear form $\langle \cdot, \cdot \rangle$ of any composition algebra $\mathbb{A}$, say with unit 1, immediately induces additional algebraic structure. First, the unit defines an embedding $\mathbb{R} \hookrightarrow \mathbb{A}$, and elements of the image of $\mathbb{R}$ are called **real**. Denote the $\langle \cdot, \cdot \rangle$-orthogonal of $\mathbb{R}$, namely the set $\{A \in \mathbb{A} : \langle A, 1 \rangle = 0\}$, by $\operatorname{Im} \mathbb{A}$. By hypothesis $\langle A, A \rangle = \langle A, A \rangle \langle 1, 1 \rangle$ for all a, so since $\langle \cdot, \cdot \rangle$ is nondegenerate, $\langle 1, 1 \rangle = 1$, and thus $\mathbb{R} \cap \operatorname{Im} \mathbb{A} = \{0\}$. Define the projections with respect to the decomposition $\mathbb{A} = \mathbb{R} \oplus \operatorname{Im} \mathbb{A}$ by $\operatorname{Re} : \mathbb{A} \to \mathbb{R}$ and $\operatorname{Im} : \mathbb{A} \to \operatorname{Im} \mathbb{A}$. We call the images $\operatorname{Re} A$ and $\operatorname{Im} A$ of an element $A \in \mathbb{A}$ respectively the **real part** and **imaginary part** of A. Denote by $\bar{\cdot}$ the linear **conjugation** map $\mathbb{A} \to \mathbb{A}$ defined by $\bar{A} := \operatorname{Re} A - \operatorname{Im} A$; by construction, $\mathbb{R}$ and $\operatorname{Im} \mathbb{A}$ are exactly the 1- and -1-eigenspaces of $\bar{\cdot}$. The bilinear form $\langle \cdot, \cdot \rangle$ can be recovered from the algebra multiplication and $\bar{\cdot}$ by the formula $\langle X, Y \rangle = \operatorname{Re}(X\bar{Y})$ (where we now view Re as the projection onto the 1-eigenspace with respect to the decomposition of $\mathbb{A}$ into the eigenspaces of $\bar{\cdot}$). Restricting $\langle X, Y \rangle$ to $\operatorname{Im} \mathbb{A}$ defines a nondegenerate bilinear form on that space. The map

$$\cdot \times \cdot : \operatorname{Im} \mathbb{A} \times \operatorname{Im} \mathbb{A} \to \operatorname{Im} \mathbb{A}$$

defined by

$$X \times Y := \operatorname{Im}(X\bar{Y})$$

is called the **cross product** on $\mathbb{A}$. Regarding $\times$ as a $(2,1)$-tensor on $\mathbb{A}$, dualizing with $\langle \cdot, \cdot \rangle$, and reversing the sign (to agree with a convention used elsewhere, including in [Sag06]) yields a 3-form $\Phi \in \Lambda^3(\operatorname{Im} \mathbb{A})^*$ defined by

$$\Phi(X, Y, Z) := -\langle X \times Y, Z \rangle = \langle XY, Z \rangle .$$

There are exactly seven composition algebras up to isomorphism. Four of the algebras are the familiar normed division algebras, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$, all endowed with the usual multiplication and conjugation, and each can be derived from the previous one using the so-called Cayley construction. The remaining three are split versions of the last three of the above, and can be described in a uniform way using an analogue of the Cayley construction: Taking $\mathbb{K}$ to be $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and endowing the set $\mathbb{K} \times \mathbb{K}$ with the multiplication $(a, b)(c, d) = (ac + d\bar{b}, \bar{a}d + cb)$ and conjugation $\overline{(a, b)} = (\bar{a}, -b)$ produces composition algebras respectively denoted $\widetilde{\mathbb{C}} \cong \mathbb{R} \oplus \mathbb{R}$, $\widetilde{\mathbb{H}} \cong M_{2\times 2}(\mathbb{R})$, and $\widetilde{\mathbb{O}}$ (here, $\cong$ denotes an algebra automorphism).

We herein restrict our attention to the $\widetilde{\mathbb{O}}$, the **split octonions**. We show that one can recover the product on $\widetilde{\mathbb{O}}$ from the imaginary split octonion cross product, $\times$, and equivalently from the corresponding 3-form $\Phi \in \Lambda^3(\operatorname{Im}\widetilde{\mathbb{O}})^*$. For $X, Y \in \operatorname{Im}\widetilde{\mathbb{O}} \subset \widetilde{\mathbb{O}}$ we have

$$XY = \operatorname{Re}(XY) + \operatorname{Im}(XY) = \left\langle X, \overline{Y} \right\rangle + X \times \overline{Y} = -\left\langle X, Y \right\rangle - X \times Y.$$

We can express $\langle \cdot, \cdot \rangle$ in terms of $\times$ via

$$\langle X, Y \rangle = -\frac{1}{6} \operatorname{tr}(X \times (Y \times \cdot)), \tag{1.43}$$

where $X \times (Y \times \cdot)$ denotes the map $\operatorname{Im}\widetilde{\mathbb{O}} \times \operatorname{Im}\widetilde{\mathbb{O}}$ [Bae02]. This fact already suffices for our purposes, but we can easily recover the product for all pairs of arguments in $\widetilde{\mathbb{O}}$, not just those in $\operatorname{Im}\widetilde{\mathbb{O}}$: Explicitly, for $(a, X), (b, Y) \in \mathbb{R} \oplus \operatorname{Im}\widetilde{\mathbb{O}} \cong \widetilde{\mathbb{O}}$, the product is given by

$$(a, X)(b, Y) = (ab - \langle X, Y \rangle, aY + bX - X \times Y).$$

Definition 1.4.2. The group G_2 is the Lie group $\operatorname{Aut}(\widetilde{\mathbb{O}})$ of algebra automorphisms of the split octonions.

Because the algebra structure of $\widetilde{\mathbb{O}}$ is equivalent to the associated 3-form $\Phi \in \Lambda^3(\operatorname{Im}\widetilde{\mathbb{O}})^*$, G_2 is also precisely the stabilizer of that 3-form in $\mathrm{GL}(\operatorname{Im}\widetilde{\mathbb{O}})$.

Remark 1.4.3. We justify the name G_2 below by showing that the Lie algebra $\mathfrak{der}(\widetilde{\mathbb{O}})$ of $\operatorname{Aut}(\widetilde{\mathbb{O}})$ is a simple real Lie algebra whose complexification has Dynkin diagram of type G_2. [FH]

Remark 1.4.4. Especially when discussed in the context of related groups also sometimes called G_2, the group just defined is sometimes denoted G_2^s, $\mathrm{G}^*_{2(2)}$, or $\widetilde{\mathrm{G}}_2$.

The group G_2 is connected and its fundamental group is $\mathbb{Z}_2$. In particular, it has a unique, simply connected, 2-fold cover; that cover admits no other quotients (up to isomorphism). There are two other groups, both related to these, sometimes called G_2. Complexifying G_2 produces the third of these groups, $\mathrm{G}_2^{\mathbb{C}}$. The other is the compact real form G_2^c of $\mathrm{G}_2^{\mathbb{C}}$; it is the maximal compact subgroup of $\mathrm{G}_2^{\mathbb{C}}$ and is the algebra automorphism group of the (standard) octonions.

In general, a k-form ϕ on a vector space V is said to be **generic** if its $\mathrm{GL}(V)$-orbit $\mathrm{GL}(V) \cdot \phi$ is open. Engel showed [Eng00] that $\mathbb{C}^7$ admits exactly one orbit of generic complex 3-forms, and that the stabilizer in $\mathrm{GL}(7, \mathbb{C})$ of any form in the orbit is isomorphic to the complexification $G_2^{\mathbb{C}}$ of G_2. The intersection of this orbit with $\Lambda^3(\mathbb{R}^7)^*$ is the union of two distinct $GL(7,\mathbb{R})$-orbits of real 3-forms. One can show that $\Phi \in \Lambda^3(\operatorname{Im}\widetilde{\mathbb{O}})^* \cong \Lambda^3(\mathbb{R}^7)^*$ is generic, so the stabilizer of an arbitrary 3-form in its orbit is isomorphic to G_2; for any 7-dimensional real vector space V, a 3-form $\phi \in \Lambda^3 V^*$ is said to be of **split type** if for some (equivalently, every) isomorphism $V^* \cong (\mathbb{R}^7)^*$ the image of ϕ under the induced isomorphism is in that orbit. The stabilizer of any 3-form in the remaining orbit is isomorphic to the compact group G_2^c, which can be realized as the algebra automorphism group of the (standard) octonions, $\mathbb{O}$.

We can alternately recover the bilinear form $\langle\cdot,\cdot\rangle$ on $\operatorname{Im}\widetilde{\mathbb{O}}$ from Φ as follows. Any 3-form ψ on a 7-dimensional real vector space V defines a symmetric bilinear form $\beta_\psi : V \times V \to \Lambda^7 V^*$ by

$$\beta_\psi(X,Y) := (X \lrcorner \psi) \wedge (Y \lrcorner \psi) \wedge \psi. \tag{1.44}$$

The determinant $\det \beta_\psi$ can be regarded as an element of $S^9 \Lambda^7 V^*$, so there is a unique element $\Omega \in \Lambda^7 V^*$ such that $\Omega^9 = \det \beta_\psi$, where Ω^9 denotes the ninth symmetric power of Ω. If ϕ is generic, then β_ψ is nondegenerate and $\det \beta_\psi \neq 0$, inducing a Hodge star isomorphism $\mathbb{R} \leftrightarrow \Lambda^7 V^*$ defined by $a \leftrightarrow a\Omega$. Composing this isomorphism with β_ψ realizes it as a real-valued, nondegenerate, symmetric bilinear form. For $\psi = \Phi$, this agrees with (1.43) up to a nonzero constant multiple.

We now derive an explicit matrix representation of the Lie algebra $\mathfrak{g}_2 = \operatorname{Der}\widetilde{\mathbb{O}}$, largely following Sagerschnig [Sag06]. We first develop some notation to describe split octonions. One explicit realization of $\widetilde{\mathbb{O}}$ is as the vector space of formal matrices of the the form

$$\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix}$$

where $x, y \in \mathbb{R}^3$ and $\xi, \eta \in \mathbb{R}$, endowed with componentwise addition and with multiplication

$$\begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix}\begin{pmatrix} \xi' & x' \\ y' & \eta' \end{pmatrix} := \begin{pmatrix} \xi\xi' + \langle x, y'\rangle & \xi x' + \eta' x + y \wedge y' \\ \eta y' + \xi' y - x \wedge x' & \eta\eta' + \langle y, x'\rangle \end{pmatrix},$$

where $\langle\cdot,\cdot\rangle$ is the standard inner product on $\mathbb{R}^3$ and $\wedge$ denotes the standard cross product on $\mathbb{R}^3$ [Zor30].

The $(4,4)$-signature bilinear form is

$$\left\langle \begin{pmatrix} \xi & x \\ y & \eta \end{pmatrix}, \begin{pmatrix} \xi' & x' \\ y' & \eta' \end{pmatrix} \right\rangle = \tfrac{1}{2}(\xi'\eta + \eta'\xi - \langle x, y' \rangle - \langle x', y \rangle).$$

The unit is

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Any algebra automorphism of a composition algebra preserves its inner product [SV00]; so, since any element of G_2 (algebra automorphism of $\widetilde{\mathbb{O}}$) fixes the unit E, it must also preserve its orthogonal complement, the seven-dimensional vector space $\operatorname{Im}\widetilde{\mathbb{O}}$ of imaginary split octonions, yielding a faithful representation $\mathrm{G}_2 \hookrightarrow \mathrm{GL}(\operatorname{Im}\widetilde{\mathbb{O}})$. Computing explicitly, $\operatorname{Im}\widetilde{\mathbb{O}}$ comprises exactly the formal matrices in $\widetilde{\mathbb{O}}$ of the form

$$\begin{pmatrix} \xi & x \\ y & -\xi \end{pmatrix}.$$

Moreover, the restriction of $\langle \cdot, \cdot \rangle$ to $\operatorname{Im}\widetilde{\mathbb{O}}$ is a $(3,4)$-signature inner product; this inner product inherits its G_2-invariance from the original inner product, yielding a faithful representation

$$\mathrm{G}_2 \hookrightarrow \mathrm{SO}(\operatorname{Im}\widetilde{\mathbb{O}}) \cong \mathrm{SO}(3,4).$$

(An analogous argument gives that G_2^c may be embedded in $\mathrm{SO}(7)$.)

Differentiating the embedding $\mathrm{G}_2 \hookrightarrow \mathrm{SO}(\operatorname{Im}\widetilde{\mathbb{O}})$ yields a Lie algebra inclusion

$$\mathfrak{g}_2 \hookrightarrow \mathfrak{so}(\operatorname{Im}\widetilde{\mathbb{O}}) \cong \mathfrak{so}(3,4).$$

Define the basis (X_i) of $\operatorname{Im}\widetilde{\mathbb{O}}$, where

$$X_1 := \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix}, \quad X_2 := \begin{pmatrix} 0 & e_2 \\ 0 & 0 \end{pmatrix}, \quad X_3 := \begin{pmatrix} 0 & e_3 \\ 0 & 0 \end{pmatrix},$$

$$X_4 := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$X_5 := \begin{pmatrix} 0 & 0 \\ -e_1 & 0 \end{pmatrix}, \quad X_6 := \begin{pmatrix} 0 & 0 \\ -e_2 & 0 \end{pmatrix}, \quad X_7 := \begin{pmatrix} 0 & 0 \\ -e_3 & 0 \end{pmatrix}.$$

With respect to this basis, the quadratic form on $\operatorname{Im}\widetilde{\mathbb{O}}$ is

$$\frac{1}{2}\begin{pmatrix} 0 & 0 & \mathbb{I}_3 \\ 0 & -1 & 0 \\ \mathbb{I}_3 & 0 & 0 \end{pmatrix}. \tag{1.45}$$

and computing directly gives that the Lie algebra $\mathfrak{so}(\operatorname{Im}\widetilde{\mathbb{O}})$ is given by

$$\left\{ \begin{pmatrix} A & v & B \\ w^T & 0 & v^T \\ C & w & -A^T \end{pmatrix} : v, w \in \mathbb{R}^3; A \in \mathfrak{gl}(3); B, C \in \mathfrak{so}(3) \right\}.$$

We now check which matrices in $\mathfrak{so}(\operatorname{Im}\widetilde{\mathbb{O}})$ are in $\mathfrak{g}_2 = \operatorname{Der}\widetilde{\mathbb{O}}$, that is, which act as derivations of $\widetilde{\mathbb{O}}$. For example, since $X_1X_2 = X_7$, any derivation

$$M = \begin{pmatrix} A & v & B \\ w^T & 0 & v^T \\ C & w & -A^T \end{pmatrix} \in \mathfrak{g}_2$$

must satisfy

$$M \cdot X_7 = M \cdot (X_1X_2) = (M \cdot X_1)X_2 + X_1(M \cdot X_2).$$

In the above notation, this reads

$$\begin{aligned} &\begin{pmatrix} A & v & B \\ w^T & 0 & v^T \\ C & w & -A^T \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ -e_3 & 0 \end{pmatrix} \\ &\qquad = \left[\begin{pmatrix} A & v & B \\ w^T & 0 & v^T \\ C & w & -A^T \end{pmatrix} \cdot \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix} \right] \begin{pmatrix} 0 & e_2 \\ 0 & 0 \end{pmatrix} \\ &\qquad\qquad + \begin{pmatrix} 0 & e_1 \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} A & v & B \\ w^T & 0 & v^T \\ C & w & -A^T \end{pmatrix} \cdot \begin{pmatrix} 0 & e_2 \\ 0 & 0 \end{pmatrix} \right]. \end{aligned}$$

Computing, this is

$$\begin{pmatrix} \sqrt{2}v^Te_3 & Be_3 \\ A^Te_3 & -\sqrt{2}v^Te_3 \end{pmatrix} = \begin{pmatrix} e_2^TCe_1 & -(\sqrt{2}w^Te_2)e_1 + (\sqrt{2}w^Te_1)e_2 \\ -[(Ae_1)\wedge e_2 + e_1 \wedge (Ae_2)] & -e_2^TCe_1 \end{pmatrix}.$$

Comparing the $(2,1)$-entries gives $A^Te_3 = -[(Ae_1)\wedge e_2 + e_1 \wedge (Ae_2)]$; expanding this expression in coordinates gives that this condition is equivalent to $\operatorname{tr} A = 0$, that is, $A \in \mathfrak{sl}(3,\mathbb{R})$. Comparing the other entries gives that particular entries of B and C must be certain expressions in v and w; applying the above argument to all other products of pairs of basis elements X_i shows that so are the other entries of B and C. In particular,

$$B = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & w_3 & -w_2 \\ -w_3 & 0 & w_1 \\ w_2 & -w_1 & 0 \end{pmatrix} \quad \text{and} \quad C = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

(where v_i and w_i are the components of v and w, respectively), so that for $x \in \mathbb{R}^3$, $Bx = -\frac{1}{\sqrt{2}}w \wedge x$ and $Cx = \frac{1}{\sqrt{2}}v \wedge x$.

Since the conditions imposed by the multiplications of pairs of basis elements X_i exhaust the conditions for a matrix in $\mathfrak{so}(\operatorname{Im}\widetilde{\mathbb{O}})$ to act as a derivation of $\widetilde{\mathbb{O}}$, $\mathfrak{g}_2$ comprises exactly the matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & v_1 & 0 & \frac{1}{\sqrt{2}}w_3 & -\frac{1}{\sqrt{2}}w_2 \\ a_{21} & a_{22} & a_{23} & v_2 & -\frac{1}{\sqrt{2}}w_3 & 0 & \frac{1}{\sqrt{2}}w_1 \\ a_{31} & a_{32} & a_{33} & v_3 & \frac{1}{\sqrt{2}}w_2 & -\frac{1}{\sqrt{2}}w_1 & 0 \\ w_1 & w_2 & w_3 & 0 & v_1 & v_2 & v_3 \\ 0 & -\frac{1}{\sqrt{2}}v_3 & \frac{1}{\sqrt{2}}v_2 & w_1 & -a_{11} & -a_{21} & -a_{31} \\ \frac{1}{\sqrt{2}}v_3 & 0 & -\frac{1}{\sqrt{2}}v_1 & w_2 & -a_{12} & -a_{22} & -a_{32} \\ -\frac{1}{\sqrt{2}}v_2 & \frac{1}{\sqrt{2}}v_1 & 0 & w_3 & -a_{13} & -a_{23} & -a_{33} \end{pmatrix}, \tag{1.46}$$

where all entries are in $\mathbb{R}$ and $a_{11} + a_{22} + a_{33} = 0$.

We can check that the Lie algebra of matrices of this form is simple. We may choose the Cartan subalgebra $\mathfrak{h} < \mathfrak{g}_2$ to be the set of diagonal matrices in $\mathfrak{g}_2$, so that $\mathfrak{h}$ is spanned by

$e_{11} - e_{22} - e_{55} + e_{66}$ and $e_{22} - e_{33} - e_{66} + e_{77}$; here, e_{ij} is the matrix with (i, j) entry 1 and all other entries 0. Computing the roots of the adjoint representation with respect to this (or any other) Cartan subalgebra of $\mathfrak{g}_2$ shows that this Lie algebra has Dynkin diagram of type G_2, justifying the name G_2 for $\mathrm{Aut}(\widetilde{\mathbb{O}})$.

We may use the root decomposition to construct a $|3|$-grading on $\mathfrak{g}_2$: Let $\phi_i \in \mathfrak{h}^*$ be the functional that returns the (i, i) entry of the element of the Lie algebra in the above matrix representation. We may choose $\alpha = \phi_2$ (a short root) and $\alpha' = \phi_1 - \phi_2$ (a long root) to be the simple roots of the $\mathfrak{g}_2$. By definition, any root may be written as a $\mathbb{Z}$-linear combination $n\alpha + n'\alpha'$ of simple roots, so we may define a grading $(\mathfrak{g}_i) = (\mathfrak{g}_2)_i$ by declaring the root spaces with α coefficient n to be in the summand $\mathfrak{g}_n$. (Note that for $i = 2$ the abbreviated notation $\mathfrak{g}_i$ for the ith graded piece conflicts with the notation $\mathfrak{g}_2$ for the Lie algebra discussed here; context will determine which object that symbol denotes.) The identity $[L_\beta, L_\gamma] \subseteq L_{\beta+\gamma}$ for root spaces $L_\beta, L_\gamma, L_{\beta+\gamma}$ guarantees that the grading $(\mathfrak{g}_i)$ respects the bracket in the sense of condition (i) of Definition 1.1.10. Direct checking—say, by inspection of the Dynkin diagram with roots labeled by grading summand—shows that the other two conditions in that definition also hold.

In the above representation, the grading $(\mathfrak{g}_i)$ is indicated by

$$\begin{pmatrix} \mathfrak{g}_0 & \mathfrak{g}_0 & \mathfrak{g}_3 & \mathfrak{g}_1 & 0 & \mathfrak{g}_2 & \mathfrak{g}_{-1} \\ \mathfrak{g}_0 & \mathfrak{g}_0 & \mathfrak{g}_3 & \mathfrak{g}_1 & \mathfrak{g}_2 & 0 & \mathfrak{g}_{-1} \\ \mathfrak{g}_{-3} & \mathfrak{g}_{-3} & \mathfrak{g}_0 & \mathfrak{g}_{-2} & \mathfrak{g}_{-1} & \mathfrak{g}_{-1} & 0 \\ \mathfrak{g}_{-1} & \mathfrak{g}_{-1} & \mathfrak{g}_2 & 0 & \mathfrak{g}_1 & \mathfrak{g}_1 & \mathfrak{g}_{-2} \\ 0 & \mathfrak{g}_{-2} & \mathfrak{g}_1 & \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_0 & \mathfrak{g}_{-3} \\ \mathfrak{g}_{-2} & 0 & \mathfrak{g}_1 & \mathfrak{g}_{-1} & \mathfrak{g}_0 & \mathfrak{g}_0 & \mathfrak{g}_{-3} \\ \mathfrak{g}_1 & \mathfrak{g}_1 & 0 & \mathfrak{g}_2 & \mathfrak{g}_3 & \mathfrak{g}_3 & \mathfrak{g}_0 \end{pmatrix}. \tag{1.47}$$

1.4.2 The G_2 *cone and the induced model geometry*

The algebraic structure of the octonions naturally induces additional structure on the space of null rays (alternately, the space of null lines) in $\mathrm{Im}\,\widetilde{\mathbb{O}}$, namely, a field of tangent 2-planes satisfying a nonintegrability condition called genericity (see Example 1.1.24 and Definition 1.5.2). This space, together with the 2-plane field, comprise a model of generic 2-plane field on 5-manifolds. In this subsection, we construct that model explicitly.

Consider the imaginary split octonions $\operatorname{Im}\widetilde{\mathbb{O}}$, the (6-dimensional) null cone of the signature-$(3,4)$ pseudo-inner product $\langle X, Y\rangle = \operatorname{Re}(X\overline{Y})$, that is, the set

$$\mathcal{N} := \{X \in \operatorname{Im}\widetilde{\mathbb{O}} - \{0\} : \langle X, X\rangle = 0\},$$

and the imaginary split octonion cross product, $X \times Y = \operatorname{Im}(X\overline{Y})$, which we may regard as a totally $\langle\cdot,\cdot\rangle$-skew $(2,1)$ tensor ${\Phi_{ab}}^c$. For any $X \in \operatorname{Im}\widetilde{\mathbb{O}}$, we have that X is contained in the subspace $X^\dagger := \ker(X \times \cdot)$. Recall the (split) octonion identity

$$X \times (X \times Y) = -\langle X, X\rangle\, Y + \langle X, Y\rangle\, X$$

and henceforth specialize to $X \in \mathcal{N}$. Substituting $Y \in X^\dagger$ gives $\langle X, Y\rangle = 0$, so $X^\dagger \subset X^\perp$. Then, interchanging the roles of X and $Y \in X^\dagger$ in the identity then gives $\langle Y, Y\rangle = 0$, that is, that $X^\dagger$ is totally null, and so $X^\dagger \subset (X^\dagger)^\perp$. Denoting the span of X by $[X]$, this defines fiberwise using just the algebraic structure on $\widetilde{\mathbb{O}}$ a filtration of the tangent space $T_X\mathcal{N} \subset T_X \operatorname{Im}\widetilde{\mathbb{O}}$, which we identify with $\operatorname{Im}\widetilde{\mathbb{O}}$ in the canonical way:

$$[X] \subset X^\dagger \subset (X^\dagger)^\perp \subset X^\perp = T_X\mathcal{N}.$$

Varying X over $\mathcal{N}$, this defines a filtration of the tangent bundle of $\mathcal{N}$. Since G_2 acts transitively on $\mathcal{N}$ and by definition preserves the algebraic structure on $\widetilde{\mathbb{O}}$, it preserves the tangent bundle filtration, and each plane field in the filtration has constant rank. Computing an example shows that $\dim X^\dagger = 3$, so the fields have respective rank 1, 3, 4, and 6.

Now consider the **null ray space** $\mathcal{R}$, the 5-dimensional space of $\langle\cdot,\cdot\rangle$-null (open) rays in $\operatorname{Im}\widetilde{\mathbb{O}}$. Since $\langle\cdot,\cdot\rangle$ has signature $(3,4)$, $\mathcal{R}$ is homeomorphic to $\mathbb{S}^2 \times \mathbb{S}^3$. By construction, these rays are precisely the orbits of the natural action of $\mathbb{R}^+$ on $\mathcal{R}$, and let $\pi : \mathcal{N} \to \mathcal{R}$ denote reduction modulo this action. Fix $x \in \mathcal{R}$; for any $X \in \pi^{-1}(x)$, we may naturally identify $T_x\mathcal{R} \leftrightarrow (T_X\mathcal{N})/[X]$, and so the above filtration of $T_X\mathcal{N}$ descends to a filtration of $T_x\mathcal{R}$. Because the filtration of $T\mathcal{N}$ is invariant under the (linear) action of $\mathbb{R}^+$, the filtration of $T_x\mathcal{R}$ is independent of the choice of preimage X, and so varying x defines a natural tangent bundle filtration,

$$\mathbb{D}_{\mathcal{R}} \subset \mathbb{D}^1_{\mathcal{R}} \subset T\mathcal{R}.$$

Checking directly in an arbitrary local frame of $\mathbb{D}_{\mathcal{R}}$ and using the transitivity of G_2 (whose action on $\mathcal{N}$ is linear and which thus descends to an action on $\mathcal{R}$) shows that (globally) $\mathbb{D}^1_{\mathcal{R}} = [\mathbb{D}_{\mathcal{R}}, \mathbb{D}_{\mathcal{R}}]$ and $TM = [\mathbb{D}_{\mathcal{R}}, [\mathbb{D}_{\mathcal{R}}, \mathbb{D}_{\mathcal{R}}]]$, and so $\mathbb{D}_{\mathcal{R}}$ is generic (again, see Example 1.1.24 and Definition 1.5.2). We call the pair $(\mathcal{R}, \mathbb{D}_{\mathbb{R}})$ the **orientable model** for the geometry of generic 2-plane fields on 5-manifolds.

Now, define P to be the stabilizer of an arbitrary null ray in $\mathcal{R}$; since G_2 acts transitively on $\mathcal{R}$, the isomorphism type of P is independent of this choice. If we choose the ray to be the one containing X_7 (defined in Subsection 1.4.1), checking directly shows that the Lie algebra $\mathfrak{p}$ of P is just $\mathfrak{g}^0 = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$, so P is indeed parabolic.

The following theorem of Sagerschnig shows that the orientable model $(\mathcal{R}, P)$ is just the model geometry $(\mathrm{G}_2 \to \mathrm{G}_2/P, \omega_{MC})$ of the parabolic geometry of type $(\mathfrak{g}_2, P)$.

Theorem 1.4.5. *[Sag06] The action of* G_2 *on* $\mathcal{R}$ *induces a diffeomorphism* $\mathrm{G}_2/P \cong \mathcal{R}$. *The tangent map of this diffeomorphism maps the* 2*-plane field* $\mathrm{G}_2 \times_P (\mathfrak{g}^{-1}/\mathfrak{p})$ *onto the field* $\mathbb{D}_{\mathcal{R}} \subset T\mathcal{R}$ *defined in Subsection 1.4.2.*

Proof. We first establish the diffeomorphisms. It is convenient to realize P concretely as the stabilizer of the null ray x_7 containing X_7, the (highest-weight) vector, which was defined in Subsection 1.4.1. Since G_2/P and $\mathcal{R}$ both have dimension 5, the orbit of x_7 is open in $\mathcal{R}$. It turns out that any model of a parabolic geometry is compact, so the map $\mathrm{G}_2 \to \mathcal{N}$ defined by $g \mapsto g \cdot X_7$ yields a diffeomorphism $\mathrm{G}_2/P \stackrel{\cong}{\to} \mathcal{R}$, and in particular, we may regard the latter as a model space for $(\mathfrak{g}_2, P)$ geometry.

We now compute each tangent space $T_x\mathcal{R}$: Recall from Subsection 1.4.2 that for any $X \in x$ we may identify $T_x\mathcal{R} = X^{\perp}/(\mathbb{R}X)$. Because both plane fields are G_2-invariant, to show that the tangent map of the diffeomorphism maps one plane field to the other, it suffices to show that it does so at any one point. We do so for the highest weight vector X_7 given in Subsection 1.4.1.

By definition, the subset $X_7^{\dagger} \subset T_{X_7}\mathcal{N}$ consists of the vectors $Y \in \operatorname{Im} \widetilde{\mathbb{O}}$ such that $X_7 Y$ is a real multiple of E. Writing

$$Y = \begin{pmatrix} \xi & x \\ y & -\xi \end{pmatrix}$$

and expanding $X_7 Y$ in formal matrices shows that this is true only when $\xi = 0$, y is a real multiple of e_3, and $x \perp e_3$. Comparing these conditions with the weight decomposition of the standard representation $V = \operatorname{Im} \widetilde{\mathbb{O}}$ of G_2 shows that $X_7^\dagger$ is a direct sum of weight spaces

$$V_{\alpha_1 + 2\alpha_2} \oplus V_{\alpha_1 + \alpha_2} \oplus V_{\alpha_2}.$$

(The first factor is the highest weight space, that is, $\mathbb{R}X_7$.) Passing to quotients yields a tangent space isomorphism $T_{\mathrm{id} \cdot P}(\mathrm{G}_2/P) \to T_{x_7}\mathcal{R}$ (alternately, $\mathfrak{g}_2/\mathfrak{p} \to X_7^\perp/(\mathbb{R}X_7)$) induced by the diffeomorphism. By construction, the filtration component $\mathfrak{g}^{-1}$ may be decomposed as $\mathfrak{p} \oplus V_{-\alpha_1} \oplus V_{-\alpha_1 - \alpha_2}$. Moreover, $\mathfrak{p}$ stabilizes X_7 and the other two factors respectively map $\mathbb{R}X_7$ onto the weight spaces $V_{\alpha_1+\alpha_2}$ and V_{α_2}. Passing to the quotient, the isomorphism maps $\mathfrak{g}^{-1}/\mathfrak{p}$ onto $X_7^\perp/(\mathbb{R}X_7)$. $\square$

Recall that there is a natural inclusion $\mathrm{G}_2 \hookrightarrow \mathrm{SO}(3,4)$. By construction, $P < \mathrm{G}_2$ coincides with the intersection $\mathrm{G}_2 \cap \dot{P}$, where $\dot{P}$ is the stabilizer in $\mathrm{O}(3,4)$ of a null ray. Nurowski exploited this inclusion to associate to any generic 2-plane field on a 5-manifold a canonical conformal structure on that manifold [Nur05]; see Subsection 1.5.3.

We henceforth use an alternate representation of G_2 and $\mathfrak{g}_2$, derived as above but using the 3-form

$$\Phi = 6dx^{012} + \sqrt{3}(dx^{036} - dx^{135} + dx^{234}) + dx^{456}, \tag{1.48}$$

on $\mathbb{R}^7$ with coordinates $x^0, \ldots, x^6$, instead of the one produced using the above construction of the split octonions. (Here, $dx^{abc} := dx^a \wedge dx^b \wedge dx^c$.) In this representation, $\mathfrak{g}_2$ comprises the matrices of the form

$$\begin{pmatrix} -(a_1+a_4) & a_8 & a_9 & -\frac{1}{\sqrt{3}}a_7 & \frac{1}{2\sqrt{3}}a_5 & \frac{1}{2\sqrt{3}}a_6 & 0 \\ b_1 & a_1 & a_2 & \frac{1}{\sqrt{3}}b_4 & -\frac{1}{2\sqrt{3}}b_3 & 0 & \frac{1}{2\sqrt{3}}a_6 \\ b_2 & a_3 & a_4 & \frac{1}{\sqrt{3}}b_5 & 0 & -\frac{1}{2\sqrt{3}}b_3 & -\frac{1}{2\sqrt{3}}a_5 \\ b_3 & a_5 & a_6 & 0 & \frac{1}{\sqrt{3}}b_5 & -\frac{1}{\sqrt{3}}b_4 & -\frac{1}{\sqrt{3}}a_7 \\ b_4 & a_7 & 0 & a_6 & -a_4 & a_2 & -a_9 \\ b_5 & 0 & a_7 & -a_5 & a_3 & -a_1 & a_8 \\ 0 & b_5 & -b_4 & b_3 & -b_2 & b_1 & a_1+a_4 \end{pmatrix}, \tag{1.49}$$

where the coefficients a_i and b_j vary over $\mathbb{R}$.

Now, the induced nondegenerate, symmetric bilinear form (1.44) is given up to a constant with respect to (dx^i) by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & h & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where

$$h = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{1.50}$$

We now fix a representations $\mathfrak{p}$: Take P to be the stabilizer in G_2 of the null ray $\partial_{x^0} \cdot \mathbb{R}^+$ in $\mathbb{R}^7$; directly computing gives that its Lie subalgebra $\mathfrak{p}$ comprises the matrices in $\mathfrak{g}_2$ of the form

$$\begin{pmatrix} -(a_1 + a_4) & a_8 & a_9 & -\frac{1}{\sqrt{3}}a_7 & \frac{1}{2\sqrt{3}}a_5 & \frac{1}{2\sqrt{3}}a_6 & 0 \\ 0 & a_1 & a_2 & 0 & 0 & 0 & \frac{1}{2\sqrt{3}}a_6 \\ 0 & a_3 & a_4 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}}a_5 \\ 0 & a_5 & a_6 & 0 & 0 & 0 & -\frac{1}{\sqrt{3}}a_7 \\ 0 & a_7 & 0 & a_6 & -a_4 & a_2 & -a_9 \\ 0 & 0 & a_7 & -a_5 & a_3 & -a_1 & a_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_1 + a_4 \end{pmatrix}, \tag{1.51}$$

where all parameters a_i are real.

Using the corresponding $|3|$-grading, one can identify the decomposition of the parabolic subalgebra $\mathfrak{p}$ into a direct sum of semisimple, abelian, and nilpotent pieces, $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, and then the corresponding decomposition $P = MAN$: Explicitly, we have

$$M = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} : B \in \mathrm{SL}(2, \mathbb{R}) \right\}$$

$$A = \left\{ \begin{pmatrix} s^{-2} & 0 & 0 & 0 & 0 \\ 0 & s\mathbb{I}_2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & s^{-1}\mathbb{I}_2 & 0 \\ 0 & 0 & 0 & 0 & s^2 \end{pmatrix} : s \in \mathbb{R}^+ \right\}$$

$$N = \left\{ \begin{pmatrix} 1 & a_8 - \frac{1}{2\sqrt{3}}a_5a_7 & a_9 - \frac{1}{2\sqrt{3}}a_6a_7 & -\frac{1}{3}a_7 & \frac{1}{2\sqrt{3}}a_5 & \frac{1}{2\sqrt{3}}a_6 & * \\ 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{2\sqrt{3}}a_6 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}}a_5 \\ 0 & a_5 & a_6 & 1 & 0 & 0 & -\frac{1}{\sqrt{3}}a_7 \\ 0 & a_7 + a_5a_6 & a_6^2 & a_6 & 1 & 0 & -a_9 - \frac{1}{2\sqrt{3}}a_6a_7 \\ 0 & -a_5^2 & a_7 - a_5a_6 & -a_5 & 0 & 1 & a_8 + \frac{1}{2\sqrt{3}}a_5a_7 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \right\}, \tag{1.52}$$

where the entry $*$ is given by

$$\frac{1}{\sqrt{3}}a_6a_8 - \frac{1}{\sqrt{3}}a_5a_9 + \frac{1}{3}a_7^2$$

and the parameters a_5, a_6, a_7, a_8, and a_9 vary over $\mathbb{R}$. In particular, the Levi subgroup G_0 of P is $MA \cong \mathrm{GL}_+(2, \mathbb{R})$.

1.5 Generic 2-plane fields on 5-manifolds

The geometry of 2-plane fields on 5-manifolds enjoys several motivations for its investigation. First, they arise naturally from a simple physical system: The configuration space of the system of two suitable surfaces rolling along one another is a 5-manifold, and the physical no-slip and no-spin conditions together define on that space a 2-plane field to which any trajectory of the system satisfying those conditions must be tangent [BM09]. Next, they are of significant historical interest: Cartan used his equivalence method to study these structures in the his famous "five-variables paper" [Car10]. Earlier, one of the first geometric realizations of an exceptional Lie algebra was that of $\mathfrak{g}_2^{\mathbb{C}}$ as the Lie algebra of vector fields preserving a particular complex generic 2-plane field, due independently to both Cartan [Car93] and Engel [Eng93]. This construction and, relatedly, that of the oriented model in Subsection 1.4.2, suggests that 2-plane fields on 5-manifolds are intimately related to various exceptional objects, including the algebra of split octonions, $\widetilde{\mathbb{O}}$, its 7-dimensional cross product (or equivalently, its defining 3-form), and the exceptional Lie groups G_2 and $\mathrm{G}_2^{\mathbb{C}}$. We exploit these relationships in Section 3.1 to construct metrics of holonomy contained in, and generically equal to, G_2.

1.5.1 Geometry of generic 2-plane fields on 5-manifolds

Generic 2-plane fields on 5-manifolds enjoy a rich intrinsic geometry.

Definition 1.5.1. The Lie bracket of two plane fields A and B on a manifold M is the subset

$$\{(x, [X, Y]_x) : X \in \Gamma(A), Y \in \Gamma(B)\} \subset TM. \tag{1.53}$$

In general this subset need not be a (constant-rank) vector subbundle, though it always will be for the objects we consider.

Definition 1.5.2. A 2-plane field $\mathbb{D}$ on a 5-manifold M is **generic** if $[\mathbb{D}, [\mathbb{D}, \mathbb{D}]] = TM$.

Given such a plane field $\mathbb{D}$ on a manifold M, we call M the **base manifold** of $\mathbb{D}$.

Recall that oriented fields of this type are in bijective correspondence with (normal, regular) parabolic geometries of type $(\mathfrak{g}_2, P)$, where P is the subgroup defined in Section 1.4. We specialize the definitions in Subsection 1.1.2: Two such fields, say, $(M_1, \mathbb{D}_1)$ and $(M_2, \mathbb{D}_2)$, are **locally equivalent** near points x_1 and x_2 (in the sense of Cartan geometries) if there is a local diffeomorphism ψ from a neighborhood U_1 of x_1 to one U_2 of x_2 such that $T\psi \cdot (\mathbb{D}_1|_{U_1}) = \mathbb{D}_2|_{U_2}$; a field is **locally flat** if it is locally equivalent near any point in the underlying manifold to the oriented model $(\mathcal{R}, \mathbb{D}_{\mathcal{R}})$ at any (equivalently, every) point.

We collect some facts here about these fields: Fix M and $\mathbb{D}$. Using the Leibniz rule for Lie brackets of vector fields shows that $[\mathbb{D}, \mathbb{D}] = \mathbb{D} + [\mathbb{D}, \mathbb{D}]$. So, if $\mathbb{D}$ is generic, then $[\mathbb{D}, \mathbb{D}]$ necessarily has constant rank 3. By construction, any local frame (X, Y) of $\mathbb{D}$ induces frames

$$(X, Y, [X, Y])$$

of $[\mathbb{D}, \mathbb{D}]$ and

$$(X, Y, [X, Y], [X, [X, Y]], [Y, [X, Y]])$$

of TM.

The field $\mathbb{D}$ defines a filtered manifold structure $(T^k M)$ on TM (see Subsection 1.1.3): Define

$$\begin{cases} T^{-1}M = \mathbb{D} \\ T^{-2}M = [\mathbb{D}, \mathbb{D}] \\ T^{-3}M = TM \end{cases}.$$

The only nontrivial summands of the induced graded bundle, $\mathrm{gr}(TM)$, are

$$\begin{cases} \mathrm{gr}_{-1}(TM) = \mathbb{D} \\ \mathrm{gr}_{-2}(TM) = [\mathbb{D}, \mathbb{D}]/\mathbb{D} \\ \mathrm{gr}_{-3}(TM) = TM/[\mathbb{D}, \mathbb{D}] \end{cases}.$$

Since $(T^k M)$ by construction respects the Lie bracket in the sense of Subsection 1.1.3, that bracket induces a Levi bracket

$$\mathcal{L} : \mathrm{gr}(TM) \times \mathrm{gr}(TM) \to \mathrm{gr}(TM).$$

This map has, up to symmetry of arguments, two nontrivial components, namely,

$$\begin{cases} \mathrm{gr}_{-1} \times \mathrm{gr}_{-1} \to \mathrm{gr}_{-2} \\ \mathrm{gr}_{-1} \times \mathrm{gr}_{-2} \to \mathrm{gr}_{-3}\,. \end{cases}.$$

Since $\mathcal{L}$ is skew-symmetric, the first descends to a map

$$\Lambda^2 \mathbb{D} \to [\mathbb{D}, \mathbb{D}]/\mathbb{D}$$

and the second is a map

$$\mathbb{D} \otimes ([\mathbb{D}, \mathbb{D}]/\mathbb{D}) \to TM/[\mathbb{D}, \mathbb{D}].$$

For both of these, genericity guarantees that the domain and codomain have the same dimension, so both are vector bundle isomorphisms.

We say that $\mathbb{D}$ is **orientable** if it is orientable as a bundle, that is, if $\Lambda^2 \mathbb{D}$ admits a nonvanishing global section. (Using the first Levi bracket isomorphism above shows that this is equivalent to $[\mathbb{D}, \mathbb{D}]/\mathbb{D}$ admitting a nonvanishing global section.)

Proposition 1.5.3. *A generic 2-plane field $\mathbb{D}$ on a 5-manifold M is orientable iff M is, and an orientation on either $\mathbb{D}$ or TM determines an orientation on the other.*

This proposition is stated in [HS09] but not proven there.

Proof. Consider a local frame (X, Y) spanning the restriction of $\mathbb{D}$ to some open set $U \subseteq M$. Since $\mathbb{D}$ is generic,

$$\Psi_U(X, Y) := X \wedge Y \wedge [X, Y] \wedge [X, [X, Y]] \wedge [Y, [X, Y]] \in \Gamma(\Lambda^5 TM|_U)$$

vanishes nowhere and so determines an orientation on U. Any other frame of $\mathbb{D}|_U$ can be written as $(aX + bY, cX + dY)$ for some

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in C^\infty(U, \mathrm{GL}(2, \mathbb{R})),$$

and computing directly shows that

$$\Psi_U(aX + bY, cX + dY) = \left[\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]^5 \Psi_U(X, Y).$$

So, if two frames determine the same orientation on $\mathbb{D}|_U$, so that the determinant of the matrix function relating them is everywhere positive, then their images under Ψ_U determine the same orientation on U, that is, Ψ_U descends to a map (which we also denote Ψ_U) from orientations on $\mathbb{D}|_U$ to orientations on U. Moreover, if two frames determine opposite orientations, so that the determinant is instead everywhere negative, their images determine opposite orientations on U, and hence Ψ_U is bijective.

Now, by construction, for any collection of open subsets $U_i \subseteq M$ satisfying the hypotheses of U above, the maps Ψ_{U_i} respectively send orientations of $\mathbb{D}|_{U_i}$ that agree on their overlaps to orientations of U_i that agree on their overlaps. In particular, if the full 2-plane field $\mathbb{D}$ is orientable, cover M with subsets U_i; the induced orientations on $\mathbb{D}|_{U_i}$ agree on their overlaps, and so their respective images under Ψ_{U_i} together define an orientation on M. Conversely, applying this argument instead to the respective images of orientations on U_i under $\Psi_{U_i}^{-1}$ shows that if M is orientable, then so is $\mathbb{D}$. □

Recall that generic 2-plane fields on oriented 5-manifolds $\mathbb{D}$ can be encoded uniquely as (normal, regular) parabolic geometries (E, ω) of type $(\mathfrak{g}_2, P)$. In the representation of $\mathfrak{g}_2$ given by (1.49), we may write the Cartan connection ω as

$$\begin{pmatrix} -(\varphi^1+\varphi^4) & \varphi^8 & \varphi^9 & -\frac{1}{\sqrt{3}}\varphi^7 & \frac{1}{2\sqrt{3}}\varphi^5 & \frac{1}{2\sqrt{3}}\varphi^6 & 0 \\ \theta^1 & \varphi^1 & \varphi^2 & \frac{1}{\sqrt{3}}\theta^4 & -\frac{1}{2\sqrt{3}}\theta^3 & 0 & \frac{1}{2\sqrt{3}}\varphi^6 \\ \theta^2 & \varphi^3 & \varphi^4 & \frac{1}{\sqrt{3}}\theta^5 & 0 & -\frac{1}{2\sqrt{3}}\theta^3 & -\frac{1}{2\sqrt{3}}\varphi^5 \\ \theta^3 & \varphi^5 & \varphi^6 & 0 & \frac{1}{\sqrt{3}}\theta^5 & -\frac{1}{\sqrt{3}}\theta^4 & -\frac{1}{\sqrt{3}}\varphi^7 \\ \theta^4 & \varphi^7 & 0 & \varphi^6 & -\theta^4 & \varphi^2 & -\varphi^9 \\ \theta^5 & 0 & \varphi^7 & -\varphi^5 & \theta^3 & -\varphi^1 & \varphi^8 \\ 0 & \theta^5 & -\theta^4 & \theta^3 & -\theta^2 & \theta^1 & \varphi^1+\varphi^4 \end{pmatrix}. \tag{1.54}$$

The grading on $\mathfrak{g}_2$ shows that the partial frame (θ^a), $a \in \{1, \ldots, 5\}$ pulls back via an arbitrary local section of $E \to M$ to a local frame on M, which we denote $(\bar{\theta}^a)$, and it is adapted to $\mathbb{D}$ in the sense that

$$\mathbb{D} = \ker\{\bar{\theta}^1, \bar{\theta}^2, \bar{\theta}^3\} \qquad \text{and} \qquad [\mathbb{D}, \mathbb{D}] = \ker\{\bar{\theta}^1, \bar{\theta}^2\}.$$

The curvature $\Omega = d\omega + \omega \wedge \omega$ of the Cartan connection ω has the form

$$\begin{pmatrix} 0 & \Phi^8 & \Phi^9 & \frac{1}{\sqrt{3}}\Phi^7 & \frac{1}{2\sqrt{3}}\Phi^5 & \frac{1}{2\sqrt{3}}\Phi^6 & 0 \\ 0 & \Phi^1 & \Phi^2 & 0 & 0 & 0 & \frac{1}{2\sqrt{3}}\Phi^6 \\ 0 & -\Phi^3 & -\Phi^1 & 0 & 0 & 0 & -\frac{1}{2\sqrt{3}}\Phi^5 \\ 0 & \Phi^5 & \Phi^6 & 0 & 0 & 0 & \frac{1}{\sqrt{3}}\Phi^7 \\ 0 & -\Phi^7 & 0 & \Phi^6 & \Phi^1 & \Phi^2 & -\Phi^9 \\ 0 & 0 & -\Phi^7 & -\Phi^5 & -\Phi^3 & -\Phi^1 & \Phi^8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.55}$$

where

$$\begin{pmatrix} \Phi^1 \\ \Phi^2 \\ \Phi^3 \\ \Phi^5 \\ \Phi^6 \\ \Phi^7 \end{pmatrix} = \begin{pmatrix} C_2 & B_2 & B_3 & A_2 & A_3 & A_3 & A_4 \\ C_3 & B_3 & B_4 & A_3 & A_4 & A_4 & A_5 \\ C_1 & B_1 & B_2 & A_1 & A_2 & A_2 & A_3 \\ D_1 & 2C_1 & 2C_2 & B_1 & B_2 & B_2 & B_3 \\ D_2 & 2C_2 & 2C_3 & B_2 & B_2 & B_3 & B_4 \\ E & D_1 & D_2 & C_1 & C_2 & C_2 & C_3 \end{pmatrix} \begin{pmatrix} \theta^1 \wedge \theta^2 \\ \theta^1 \wedge \theta^3 \\ \theta^2 \wedge \theta^3 \\ \theta^1 \wedge \theta^4 \\ \theta^1 \wedge \theta^5 \\ \theta^2 \wedge \theta^4 \\ \theta^2 \wedge \theta^5 \end{pmatrix} \tag{1.56}$$

and Φ_8 and Φ_9 satisfy additional equations that we will not need. (This is taken from [Nur05], though some of the matrix components of the representation of $\mathfrak{g}_2$ there differ from those in the representation (1.49) by constant factors, as do the coefficients A_i, B_i, C_i, D_i, E in (1.56).)

1.5.2 ODE realization of the geometry and a quasi-normal form

Consider ordinary differential equations of the form

$$z' = F(x, y, y', y'', z), \tag{1.57}$$

where F is some function on an open subset of $\mathbb{R}^5$, and y and z are functions of x. We may realize such equations geometrically using the standard machinery of the geometry of differential equations as follows [Nur05]: First denote $p := y'$ and $q := y''$, so that $\mathbb{R}^5$ denotes $xypqz$-space. For any smooth functions y and z, the graph of the prolongation (x, y, y', y'', z) is tangent everywhere to the tautological 3-plane field defined by the conditions

$$dy - p\,dx = 0$$

$$dp - q\,dx = 0.$$

Any solution to (1.57) must satisfy an additional equation determined by F; explicitly, the prolonged solution $(x(t), y(t), y'(t), y''(t), z(t))$ must be tangent to the field $\mathbb{D}_F$ defined by

$$dz - F\,dx = 0$$

$$dy - p\,dx = 0$$

$$dp - q\,dx = 0.$$

The 1-forms on the left-hand sides are linearly independent, so $\mathbb{D}_F$ has constant rank 2. The vector fields ∂_q and $D := \partial_x + p\partial_y + q\partial_p + F\partial_z$ are linearly independent and annihilate those 1-forms, so together they span $\mathbb{D}_F$ everywhere.

Now, computing gives

$$[\partial_q, D] = \partial_p + F_q\partial_z,$$

which is annihilated by the 1-form $dy - p\,dx$ and the linear combination

$$(dz - F\,dx) - F_q(dp - q\,dx)$$

of the defining forms. So, $[\mathbb{D}, \mathbb{D}]$ is the common kernel of those two forms.

Proposition 1.5.4. *Let F be a real-valued function on an open subset of $\mathbb{R}^5$. The field $\mathbb{D}_F$ is generic iff F_{qq} is nonvanishing.*

Proof. Computing gives

$$\begin{aligned}[\partial_q, [\partial_q, D]] &= F_{qq}\partial_z \\ [D, [\partial_q, D]] &= -\partial_y + (DF_q - [\partial_q, D]F)\partial_z.\end{aligned}$$

Evaluating gives

$$(dx \wedge dy \wedge dp \wedge dq \wedge dz)(\partial_q \wedge D \wedge [\partial_q, D] \wedge [\partial_q, [\partial_q, D]] \wedge [D, [\partial_q, D]]) = -F_{qq},$$

so the set

$$\{\partial_q, D, [\partial_q, D], [\partial_q, [\partial_q, D]], [D, [\partial_q, D]]\}$$

spans TM and hence the plane field $\mathbb{D}_F$ is generic iff F_{qq} vanishes nowhere. □

Remark 1.5.5. Cartan studied exhaustively in [Car15] the ordinary differential equations (1.57) for which $F_{qq} = 0$.

The following proposition of Bryant and Hsu (following Cartan) shows that, locally, all $\mathbb{D}$ are induced by functions F as above, defining a local quasi-normal form for generic 2-plane fields on 5-manifolds. It is possible for distinct functions F and $\overline{F}$ to yield locally equivalent 2-plane fields, that is, for a pair $(F, \bar{F})$ to admit a local diffeomorphism ϕ such that $\mathbb{D}_{\bar{F}} = T\phi \cdot \mathbb{D}_F$, and so the quasi-normal form is not strictly normal.

Proposition 1.5.6. *[BH93, Section 2] Let $\mathbb{D}$ be a generic 2-plane field on a 5-manifold M. For every point $s \in M$ there are local coordinates (x, y, p, q, z) on some neighborhood U mapping s to the origin and a function $F \in \Gamma(C^\infty(U))$ such that $\mathbb{D}|_U = \mathbb{D}_F$.*

Example 1.5.7. The plane field $\mathbb{D}_F$ induced by $F(x, y, p, q, z) = q^2$ is locally equivalent to the plane fields of the orientable model in Subsection 1.4.2 [Nur05, Section 5.3].

1.5.3 Nurowski's canonical conformal structure

Recall the following construction of the model geometry of conformal geometry of signature (p, q): Let $\langle \cdot, \cdot \rangle$ be a nondegenerate, symmetric bilinear form of signature $(p + 1, q + 1)$ on $\mathbb{R}^{p+1,q+1}$, let $\mathcal{N}$ be its null cone,

$$\mathcal{N} := \{X \in \mathbb{R}^{p+1,q+1} - \{0\} : \langle X, X \rangle = 0\},$$

and let $\mathcal{R}$ be the ray space, that is, the space of null (open) rays in $\mathbb{R}^{p+1,q+1}$. The projection $\pi : \mathcal{N} \to \mathcal{R}$ and the dilations $\delta^s : \mathcal{N} \to \mathcal{N}$, $s > 0$, defined by $\delta^s(X) := sX$, realize $\mathcal{N}$ as a $\mathbb{R}^+$-principal bundle over $\mathcal{R}$. Pulling back the standard metric (the one induced via the canonical identifications $T_x\mathbb{R}^{p+1,q+1} \leftrightarrow \mathbb{R}^{p+1,q+1}$, $x \in \mathbb{R}^{p+1,q+1}$) by an arbitrary section of $g \in \Gamma(\mathcal{N} \to \mathcal{R})$ defines a metric, which we also denote g, on $\mathcal{R}$. By construction, any other section (metric) is a nonvanishing $C^\infty(\mathcal{R})$ multiple of g, so $\langle \cdot, \cdot \rangle$ induces the natural conformal structure $c_{\mathcal{R}} := [g]$ on $\mathcal{R}$. By construction $\pi : \mathcal{N} \to \mathcal{R}$ is the metric bundle of c, and δ^s coincides with the so-named dilations in Subsection 1.2.3. One can show that $(\mathcal{R}, c_{\mathcal{R}})$ coincides with the model geometry of the parabolic geometry of type $(\mathfrak{o}(p + 1, q + 1), \dot{P})$ discussed in Examples 1.1.7 and 1.1.21 [Sha97].

Now, taking $\langle \cdot, \cdot \rangle$ to be the bilinear form on $\operatorname{Im} \widetilde{\mathbb{O}}$ defined in Subsection 1.4.1 yields a conformal class on the space of the oriented model $(\mathcal{R}, \mathbb{D}_{\mathcal{R}})$ of generic 2-plane fields on 5-manifolds described in Subsection 1.4.2. Nurowski showed that, remarkably, one can generalize this construction and canonically assign to a generic 2-plane field $\mathbb{D}$ on a 5-manifold a conformal class $c_{\mathbb{D}}$ on M [Nur05]. We can see this efficiently using the language of parabolic geometries. First, assume that M is orientable; recall again that we can canonically encode such a field $\mathbb{D}$ as a (normal, regular) parabolic geometry (E, ω) of type $(\mathfrak{g}_2, P)$. Since

$P = \mathrm{G}_2 \cap \dot{P}$, where $\dot{P}$ is the stabilizer in $\mathrm{O}(3,4)$ of a null ray, G_2/P is canonically isomorphic to $\mathrm{O}(3,4)/\dot{P}$, and differentiating gives that $\mathfrak{g}_2/\mathfrak{p}$ is canonically isomorphic to $\mathfrak{o}(p+1, q+1)/\dot{\mathfrak{p}}$. Define the $\dot{P}$-principal bundle

$$\dot{E} := E \times_P \dot{P};$$

then, we may uniquely extend ω (which takes values in $\mathfrak{g}_2$) by $\dot{P}$-equivariance to an $\mathfrak{o}(p+1, q+1)$-valued form $\dot{\omega}$ on $\dot{E}$. By equivariance, $(\dot{E}, \dot{\omega})$ is a parabolic geometry of type $\mathfrak{o}(p+1, q+1, \dot{P})$ and so it defines a conformal structure on M. (One can furthermore show [HS09, Proposition 4] that $\dot{\omega}$ is in fact the normal Cartan connection associated to that structure.) Since $\dot{\omega}$ extends ω by $\dot{P}$-equivariance, the curvature $\dot{\Omega}$ of $\dot{\omega}$ extends the curvature Ω of ω by equivariance.

The construction of this conformal structure is local and does not depend on the orientation of the underlying manifold; so for a generic 2-plane field on a nonorientable 5-manifold M, we may use it to define conformal structures on the sets in an open cover of M by orientable sets, and these patch together to define a global conformal structure.

We henceforth refer to a conformal structure induced by a generic 2-plane field in this way (whether the underlying manifold is oriented or note) as a **Nurowski conformal structure**.

Consulting the bilinear form (1.50) corresponding to the representation (1.49) shows the representative metric $g \in c_{\mathbb{D}}$ determined by the section σ is just $-2\bar{\theta}^1\bar{\theta}^5 + 2\bar{\theta}^2\bar{\theta}^4 - (\bar{\theta}^3)^2$ (see (1.54) and the paragraph following it for the definition of the frame $(\bar{\theta}^a)$).

Proposition 1.5.8. *For any generic* 2*-plane field* $\mathbb{D}$*, and with respect to the induced conformal structure* $c_{\mathbb{D}}$*,* $\mathbb{D}$ *is totally null (that is,* $\mathbb{D} \subset \mathbb{D}^{\perp}$*) and* $[\mathbb{D}, \mathbb{D}] = \mathbb{D}^{\perp}$*.*

We will invoke the latter equality regularly (even without explicit reference to the induced conformal structure), and without comment.

Proof. These facts follow immediately and respectively from the above formula of the representative metric in terms of $(\bar{\theta}^a)$ and the adaptation of that coframe to $\mathbb{D}$. □

We can use the induced conformal structure of a generic 2-plane field on a 5-manifold to produce various natural isomorphisms; we use them in the construction of the Cartan curvature tensor $A \in \Gamma(S^4\mathbb{D}^*)$ (see Subsection 1.5.4 below).

Lemma 1.5.9. *Let $\mathbb{D}$ be a generic 2-plane field on a 5-manifold M. There is a natural isomorphism $\psi : (TM/\mathbb{D}^\perp)[-1] \stackrel{\cong}{\to} \mathbb{D}^*[1]$. An orientation on M induces a natural isomorphism $\mu : \mathbb{D} \stackrel{\cong}{\to} \mathbb{D}^\perp[-1]$, and composing these gives a natural isomorphism $\tau := \psi \circ \mu : \mathbb{D} \to (TM/\mathbb{D}^\perp)[-1]$.*

Proof. That $\mathbb{D}$ is totally null has two relevant consequences: First, lowering an index with $\mathbf{g}$ descends to a map $\psi : (TM/\mathbb{D}^\perp)[-1] \to \mathbb{D}^*[1]$, and by construction it is an isomorphism. Second, $\mathbf{g}$ defines a negative definite weighted fiber metric (which we also denote $\mathbf{g}$) on $\mathbb{D}^\perp/\mathbb{D}$, that is, a section of $(S^2(\mathbb{D}^\perp/\mathbb{D})^*)[2]$. If M is oriented, by Proposition 1.5.3 so is $\mathbb{D}$. Then, via the Levi bracket (defined in Subsection 1.5.1), this determines an orientation on $\mathbb{D}^\perp/\mathbb{D}$ and thus on the weighted bundle $(\mathbb{D}^\perp/\mathbb{D})[-1]$, and then lowering an index with the induced bilinear form $\mathbf{g}$ determines an orientation on $(\mathbb{D}^\perp/\mathbb{D})^*[1]$. Since the lattermost bundle has rank 1, there is a unique positively oriented section $\alpha \in \Gamma((\mathbb{D}^\perp/\mathbb{D})^*[1])$ that satisfies $\mathbf{g}^{-1}(\alpha, \alpha) = -\frac{3}{4}$. (This constant is chosen so that τ has a simple formula in terms of local frames of TM that we produce later.) Then, define the map $\mu : \mathbb{D} \stackrel{\cong}{\to} \mathbb{D}^*[1]$ by

$$\mu(X) := \alpha(\mathcal{L}(X \wedge \cdot)),$$

or equivalently,

$$\mu(X)(Y) := \alpha([X, Y] + \mathbb{D}).$$

It is an isomorphism because (again) the component $\Lambda^2\mathbb{D} \stackrel{\cong}{\to} \mathbb{D}^\perp/\mathbb{D}$ of the Levi bracket is. □

By construction, reversing the orientation of M replaces μ and τ with their negatives.

Recall from Subsection 1.5.2 that any function $F(x, y, p, q, z)$ for which F_{qq} is nonvanishing determines a generic 2-plane field $\mathbb{D}_F$ on a 5-manifold. Nurowski computed a (formidable) formula, recorded here as (A.1) in Appendix A.1, for a representative g_F of the conformal structure $c_F := c_{\mathbb{D}_F}$ it induces. In particular, g_F is polynomial in F and derivatives of F of order no more than 4, and so if F is real-analytic, so is g_F and hence so is c_F. Computing directly shows that the determinant of the matrix of g_F in the basis $(\widetilde{\omega}^a)$

(also defined in that appendix) is a real multiple of F_{qq}^{18}, and so $(g_F)^{-1}$ is polynomial in F, derivatives of F of order no more than 4, and F_{qq}^{-1}.

Example 1.5.10. [Nur05, Example 6], [Nur08b, Section 3] Suppose F depends on $q = y''$ alone, so that the corresponding differential equation is just $z' = F(y'')$. Then, c_F contains the Ricci-flat representative $e^{2\Upsilon} g_F$, where $\Upsilon(q)$ satisfies an explicit second-order ordinary differential equation.

1.5.4 Cartan fundamental curvature tensor

To any generic 2-plane field $\mathbb{D}$ on an oriented 5-manifold one can associate the curvature Ω of the corresponding normal, regular parabolic geometry (E, ω) of type $(\mathfrak{g}_2, P)$. The essential component of Ω is the *Cartan curvature tensor* $A \in \Gamma(S^4 \mathbb{D}^*)$ defined below [Car10]. Its role in the geometry of generic 2-plane fields on 5-manifolds is similar to that of Weyl curvature in conformal geometry in dimension $n \geq 4$: Among other reasons, it is precisely the obstruction to local flatness. In fact, below we derive A from the Weyl curvature of the induced conformal class $c_{\mathbb{D}}$ and extend the definition of A to fields on general 5-manifolds.

Let $\mathbb{D}$ be an oriented, generic 2-plane field on a 5-manifold, consider the induced conformal class $c_{\mathbb{D}}$ on M, and let (E, ω) and $(\dot{E}, \dot{\omega})$ respectively be the corresponding normal, regular parabolic geometries of types $(\mathfrak{g}_2, P)$ and $(\mathfrak{o}(p+1, q+1), \dot{P})$; recall that $\dot{E} = E \times_P \dot{P}$ and that $\dot{\omega}$ is just given by extending ω to $\dot{E}$ by equivariance. Coerce an arbitrary section $\sigma \in \Gamma(E)$ into a section $\dot{\sigma} \in \Gamma(\dot{E}) = \Gamma(E \times_P \dot{P})$ by setting $\dot{\sigma}(x) := [\sigma(x), \mathrm{id}]$ and let $g \in c_{\mathbb{D}}$ be the representative metric determined by $\dot{\sigma}$ as in Example 1.1.21. Now, the representation (1.49) of $\mathfrak{g}_2$ corresponds in the sense of Subsection 1.4.1 to a bilinear form on $\mathbb{R}^{3,4}$ of the form (1.15), so consulting (1.17) lets us identify the Weyl curvature W of g as the piece of the Cartan curvature $\dot{\Omega}$ taking values in $\mathfrak{o}(h)$. Since $\dot{\sigma}$ and $\dot{\Omega}$ respectively extend σ and Ω

by $\dot{P}$-equivariance, $\dot{\sigma}^*\dot{\Omega} = \sigma^*\Omega$, which is given by (1.55). Explicitly, we have

$$\frac{1}{2}\Omega_{ab}{}^c{}_d\bar{\theta}^a \wedge \bar{\theta}^b = \begin{pmatrix} \Phi_1 & \Phi_2 & 0 & 0 & 0 \\ -\Phi_3 & -\Phi_1 & 0 & 0 & 0 \\ \Phi_5 & \Phi_6 & 0 & 0 & 0 \\ -\Phi_7 & 0 & \Phi_6 & \Phi_1 & \Phi_2 \\ 0 & -\Phi_7 & -\Phi_5 & -\Phi_3 & \Phi_1 \end{pmatrix}$$

Lowering an index using g gives

$$\frac{1}{2}\Omega_{abcd}\bar{\theta}^a \wedge \bar{\theta}^b = \begin{pmatrix} 0 & \Phi_7 & \Phi_5 & \Phi_3 & \Phi_1 \\ -\Phi_7 & 0 & \Phi_6 & \Phi_1 & \Phi_2 \\ -\Phi_5 & -\Phi_6 & 0 & 0 & 0 \\ -\Phi_3 & -\Phi_1 & 0 & 0 & 0 \\ -\Phi_1 & -\Phi_2 & 0 & 0 & 0 \end{pmatrix}. \tag{1.58}$$

By construction, pulling back Ω_{abcd} via σ just gives the Weyl curvature W_{abcd} of g.

Lemma 1.5.11. *Let $\mathbb{D}$ be a generic 2-plane field on an oriented 5-manifold. The Weyl tensor of the induced conformal structure $c_{\mathbb{D}}$ satisfies $W(\cdot,\cdot,X,Y) = 0$ for all $X, Y \in \Gamma(\mathbb{D}^\perp)$.*

Proof. The field $\mathbb{D}^\perp$ is exactly the common kernel of $\bar{\theta}^1$ and $\bar{\theta}^2$; consulting (1.58) shows that $W_{abcd}\bar{\theta}^a \wedge \bar{\theta}^b = 0$ for $c, d \geq 3$, giving the claim. □

We are now prepared to define the Cartan curvature tensor A of a generic 2-plane field $\mathbb{D}$ on an 5-manifold M. First, suppose that M oriented. The Weyl curvature W has conformal weight 2, that is, we may regard it as a section of $(\bigotimes^4 T^*M)[2]$. Then, define $A \in \Gamma(\bigotimes^4 \mathbb{D}^*)$ by

$$A(Z_1, Z_2, Z_3, Z_4) := W(Z_1, \tau(Z_2), Z_3, \tau(Z_4)),$$

where $\tau : \mathbb{D} \to (TM/\mathbb{D}^\perp)[-1]$ is the isomorphism defined in Lemma 1.5.9 and where we regard $\tau(Z_2)$ and $\tau(Z_4)$ as arbitrary representatives of those cosets in $TM[-1]$. By the previous lemma, A is independent of the choice of those representatives. Since W has conformal weight 2 and each argument $\tau(Z_\bullet)$ has conformal weight -1, A has conformal weight 0 as indicated. Reversing the orientation of M replaces τ with its negative, but because τ occurs precisely twice in the definition, A is independent of the orientation of M. It moreover depends only on local data, so we may define it for $\mathbb{D}$ on nonorientable M, too, by defining it locally and patching as we did to define $c_{\mathbb{D}}$ for generic 2-plane fields $\mathbb{D}$ on nonorientable 5-manifolds.

Proposition 1.5.12. *The tensor A is totally symmetric (for every generic 2-plane field $\mathbb{D}$ on a 5-manifold M), that is, we may regard A as a section of $S^4\mathbb{D}^*$.*

Proof. Consider a local section σ of $E \to M$, let $(\bar{\theta}^a)$ be the induced pullback coframe on M, let (X_a) denote the dual frame, and let $g \in c$ denote the induced representative $-2\bar{\theta}^1\bar{\theta}^5 + 2\bar{\theta}^2\bar{\theta}^4 - (\bar{\theta}^3)^2$. Using g, we may freely identify weighted and unweighted sections. By construction, $\mathbb{D}$ is locally spanned by X_4 and X_5, and $\mathbb{D}^\perp$ is locally spanned by X_3, X_4, and X_5. Then, lowering an index gives that the isomorphism $\psi : \mathbb{D}^*[1] \to (TM/\mathbb{D}^\perp)[-1]$ in the proof of Lemma 1.5.9 is characterized by

$$\begin{cases} \psi(\theta^4|_{\mathbb{D}}) = X_2 + \mathbb{D}^\perp \\ \psi(\theta^5|_{\mathbb{D}}) = -X_1 + \mathbb{D} \end{cases},$$

where we may use g to regard unweighted sections as weighted sections of the indicated bundles. Now, since $\bar{\theta}^3$ annihilates $\mathbb{D}$, it descends to a section of $(TM/\mathbb{D})^*$, and restricting it, we may regard it as a section of $(\mathbb{D}^\perp/\mathbb{D})^*$ or $(\mathbb{D}^\perp/\mathbb{D})^*[1]$ (we denote this resulting section by $\bar{\theta}^3$ too). Then, $\frac{\sqrt{3}}{2}\bar{\theta}^3$ satisfies $\mathbf{g}^{-1}(\theta^3, \theta^3) = -\frac{3}{4}$, so for the appropriate orientation on the domain of σ, we may choose that form for α in the proof of Lemma 1.5.9. Using the definition of μ gives

$$\begin{aligned} \mu(X_4)(X_5) &= \alpha([X_4, X_5] + \mathbb{D}) \\ &= \tfrac{\sqrt{3}}{2}\bar{\theta}^3([X_4, X_5]) \\ &= \tfrac{\sqrt{3}}{2}[-d\bar{\theta}^3(X_4, X_5) + X_4(\bar{\theta}^3(X_5)) - X_5(\bar{\theta}^3(X_4))]. \end{aligned}$$

The last two terms in brackets vanish because $(\bar{\theta}^a)$ and (X_a) are dual bases. Then, using (1.54) and (1.55) and comparing the ${}^0{}_3$ components of the definition $\Omega = d\omega + \omega \wedge \omega$ gives $d\theta^3 = -\frac{2}{\sqrt{3}}\theta^4 \wedge \theta^5$, so substituting gives

$$\mu(X_4)(X_5) = -(\theta^4 \wedge \theta^5)(X_4, X_5) = -1.$$

Since μ is antisymmetric by definition, this computation completely characterizes μ, giving

$$\begin{cases} \mu(X_4) = -\theta^5|_{\mathbb{D}} \\ \mu(X_5) = \theta^4|_{\mathbb{D}} \end{cases} .$$

Then, composing gives that the isomorphism $\tau = \psi \circ \mu$ is characterized by

$$\begin{cases} \tau(X_4) = X_1 + \mathbb{D}^{\perp} \\ \tau(X_5) = X_2 + \mathbb{D}^{\perp} \end{cases} .$$

Then, checking in the basis $\{X_4, X_5\}$ of $\mathbb{D}$, which involves consulting the Weyl curvature 1.58, shows that A is fully symmetric. □

Remark 1.5.13. By construction, the coefficients of A in the frame of $S^4\mathbb{D}^*$ induced by the coframe $(\theta^4|_{\mathbb{D}}, \theta^5|_{\mathbb{D}})$ are just the coefficients $A_1, \ldots, A_5$ in (1.56).

Remark 1.5.14. One can show that if A vanishes for a given $\mathbb{D}$, then the full curvature Ω of the corresponding normal, regular parabolic geometry vanishes; so, $A = 0$ iff $\mathbb{D}$ is locally flat. [Car10]

1.6 Holonomy

1.6.1 Pseudo-Riemannian holonomy

The holonomy of a pseudo-Riemannian manifold (N, h) is a group that measures the failure of parallel transport around closed loops (by the Levi-Civita connection ∇^h of h) to preserve geometric data; for the general material in this section, we partially follow [Pet06]. For any piecewise smooth curve $\gamma : [0, 1] \to N$ based at $x \in N$, parallel transport along γ with respect to ∇^h defines a linear map $P_\gamma : T_{\gamma(0)}N \to T_{\gamma(1)}N$; if γ is moreover a loop based at x, then P_γ is a map $T_xN \to T_xN$. By construction, parallel transport $P_{\gamma^{-1}}$ along the reverse loop γ^{-1} satisfies $P_\gamma^{-1} P_\gamma = \mathrm{id}$, so $P_\gamma \in \mathrm{GL}(T_xN)$. In fact, since the metric is parallel with respect to the Levi-Civita connection by definition of the latter, P_γ preserves lengths and hence $P_\gamma \in \mathrm{O}(T_xN)$.

Definition 1.6.1. The **holonomy (group)** of the pseudo-Riemannian manifold (N, h) based at $x \in M$ is the group

$$\mathrm{Hol}_x(N, h) := \{P_\gamma : \gamma \in \Omega_x(N)\} \leq \mathrm{O}(T_xN), \tag{1.59}$$

where $\Omega_x(N)$ denotes the space of (piecewise-smooth) loops in N based at x.

Note that the holonomy is indeed a group because it is closed under composition: $P_{\gamma\cdot\gamma'} = P_{\gamma'}P_\gamma$, where $\cdot$ denotes concatenation. Picking an pseudo-orthonormal basis of a tangent space T_xN realizes $\operatorname{Hol}_x(N,h)$ as an explicit subgroup of $\mathrm{O}(p,q)$ and any other choice yields a conjugate subgroup, so without reference to bases we may regard $\operatorname{Hol}_x(N,h)$ as a conjugacy class of subgroups of $\mathrm{O}(p,q)$. Moreover, given a path α from x to y in N, we have by construction that

$$\operatorname{Hol}_y(N,h) = P_\alpha \operatorname{Hol}_x(N,h) P_{\alpha^{-1}}.$$

Picking pseudo-orthonormal bases of T_xN and T_yN and applying this identity shows that the conjugacy class of $\operatorname{Hol}_x(N,h)$ in $\mathrm{O}(p,q)$ is also independent of the base point x. So, for path-connected manifolds, we may unambiguously refer to that conjugacy class as the **holonomy** $\operatorname{Hol}(N,h)$ (or $\operatorname{Hol}(h)$) of (N,h).

Similarly, the **restricted holonomy (group)** of (N,h) based at $x \in M$ is the subgroup $\operatorname{Hol}^0(N,h) \leq \operatorname{Hol}(N,h)$ defined by

$$\operatorname{Hol}_x(N,h) := \{P_\gamma : \gamma \in \Omega_x^0(N)\} \leq \mathrm{O}(T_xN),$$

where $\Omega_x^0(N)$ denotes the space of (piecewise-smooth) *contractible* loops based at x. As above, we may identify it with a conjugacy class of subgroups in $\mathrm{O}(p,q)$ (in fact, of its identity component, $\mathrm{SO}^+(p,q)$), and this identification is independent of the base point x, so we may again suppress the base point in the notation.

Directly applying the definition of holonomy to a product manifold $(N_1,h_1)\times(N_2,h_2)$ gives the product identity

$$\operatorname{Hol}_{(x_1,x_2)}(N_1\times N_2, h_1\oplus h_2) = \operatorname{Hol}_{x_1}(N_1,h_1)\times\operatorname{Hol}_{x_2}(N_2,h_2), \tag{1.60}$$

so any factorization of a pseudo-Riemannian manifold into a Cartesian product is reflected in a decomposition of the holonomy.

Let (N,h) be a pseudo-Riemannian manifold, fix arbitrary $x \in N$, and let $T_xN = \bigoplus_k E_k$ be a decomposition of T_xN into $\operatorname{Hol}_x(N,h)$-submodules. Since parallel translation along loops based at x preserves this decomposition by definition, parallel transport of the subspaces E_k defines respective parallel plane fields η_k and thus a global decomposition

$$TM = \bigoplus \eta_k$$

into parallel plane fields. Because they are parallel, they are necessarily integrable. The following theorem says that such a decomposition can be used to factor (N, h) at least locally as a pseudo-Riemannian product manifold.

Theorem 1.6.2 (de Rham Decomposition)**.** *Let (N, h) be a pseudo-Riemannian manifold and let $TN = \bigoplus_k \eta_k$ be a decomposition of TN into indecomposable $\mathrm{Hol}_x(N, h)$-submodules. Then, any point $x \in N$ admits a neighborhood U with product structure*

$$(U, h|_U) = \left(\prod_k U_k, \sum_k h_k \right)$$

such that $TU_k = \eta_k|_{U_k}$. By construction, the holonomy group $\mathrm{Hol}_\bullet(U_k, h_k)$ of every factor acts indecomposably on $T_\bullet U_k$, respectively.

Remark 1.6.3. de Rham proved this result for definite signature [dR52] and Wu for arbitrary signature [Wu64]. The form of this theorem given here is that of [Pet06, Theorem 56] (except the statement there also points out if N is simply connected and complete, the decomposition can in fact be taken to be global). The critical difference between the definite- and indefinite-signature cases is that the action of the holonomy group on the tangent space at a point of a Riemannian manifold decomposes that space into direct sum of *irreducible* modules, whereas for a general pseudo-Riemannian manifold, not all indecomposable modules of the holonomy group need be irreducible.

The general-signature version of Berger's Theorem imposes strong restrictions on the groups that can occur as the holonomy of a simply-connected pseudo-Riemannian manifold that is not locally symmetric.

Theorem 1.6.4 (Berger's List)**.** *Let (N, h) be a simply connected pseudo-Riemannian n-manifold that is not a locally symmetric space. If $\mathrm{Hol}_x(N, h)$ acts irreducibly on $T_x N$, then up to isomorphism it is one of the following.*

- *If h has definite signature $(n, 0)$, $\mathrm{Hol}(N, h)$ is one of the following:* $\mathrm{SO}(n)$, $\mathrm{U}(\frac{n}{2})$, $\mathrm{SU}(\frac{n}{2})$, $\mathrm{Sp}(\frac{n}{4}) \cdot \mathrm{Sp}(1)$, $\mathrm{Sp}(\frac{n}{4})$, G_2^c *(see Subsection 1.4.1),* $\mathrm{Spin}(7)$.

- *If h has indefinite signature (p,q), $\mathrm{Hol}(N,h)$ is one of the following:* $\mathrm{SO}(p,q)$, $\mathrm{U}(\frac{p}{2},\frac{q}{2})$, $\mathrm{SU}(\frac{p}{2},\frac{q}{2})$, $\mathrm{Sp}(\frac{p}{4},\frac{q}{4})\cdot\mathrm{Sp}(1)$, $\mathrm{Sp}(\frac{p}{4},\frac{q}{4})$, G_2, $\mathrm{Spin}(3,4)$, $\mathrm{SO}(n,\mathbb{C})$, $\mathrm{G}_2^{\mathbb{C}}$, $\mathrm{Spin}(7,\mathbb{C})$.

Remark 1.6.5. Berger's original list contained several additional possible groups, in both the definite- and indefinite-signature cases, which have since been shown to occur only for symmetric spaces. By contrast, all groups given in the theorem as given here have been realized as the holonomy groups of concrete pseudo-Riemannian manifolds that are not symmetric spaces.

Because holonomy measures the failure of parallel transport to preserve geometric objects on a pseudo-Riemannian manifold, a manifold with special holonomy (that is, holonomy other than $\mathrm{SO}(p,q)$) necessarily admits some parallel structure.

For a Lie group $G \leq \mathrm{GL}(n,\mathbb{R})$, a G-**structure** on an n-manifold N is a reduction of TN to G, that is, a subbundle E of the frame bundle $\mathcal{F}$ of TN whose fiber E_x at $x \in N$ consists of a G-orbit of frames in $\mathcal{F}_x$. A G-structure E is **parallel** if it is invariant under parallel translation, that is, if for any $\gamma \in \Omega_x(N)$ (say, with $\gamma(1) = y$) and any $(X_a) \in E_x$ we have $(P_\gamma X_a) \in E_y$, where P_γ is parallel translation along γ.

Proposition 1.6.6. *If the holonomy of a pseudo-Riemannian manifold (N,h) of signature (p,q) is contained in some Lie group $G < \mathrm{O}(p,q)$, then h admits a parallel G-structure.*

The main application of Theorem 2.1.2 in this work regards metrics of holonomy contained in G_2 (see Subsection 1.6.3 and Section 3.1); we record some specialized results about such holonomy here.

Theorem 1.6.7 (Bonan's Theorem)**.** *[Bon66] Any manifold with holonomy contained in G_2 is Ricci-flat.*

Lemma 1.6.8. *Given a 7-dimensional pseudo-Riemannian manifold (N,h), the holonomy group $\mathrm{Hol}_\bullet(N,h)$ is contained in $\mathrm{G}_2 < \mathrm{SO}(h)$ (which requires that N be orientable) iff h admits a parallel 3-form Φ of split type compatible with the metric in the sense that, up to a constant factor*

$$(\cdot \lrcorner\, \Phi) \wedge (\cdot \lrcorner\, \Phi) \wedge \Phi = h \, \mathrm{vol}_h \, .$$

Proof. (⇒) If $\mathrm{Hol}(N,h) \leq \mathrm{G}_2 < \mathrm{SO}(h)$, then by Proposition 1.6.6 (N,h) admits a parallel G_2-structure. By the characterization of G_2 in Subsection 1.4.1, the fiber of that G_2-structure over any point $x \in M$ can be regarded as the set of frames in which the coordinate representation of some generic 3-form $\phi_x \in \Lambda^3 T_x^* M$ is given by a fixed split type (generic) 3-form on $\mathbb{R}^7$ compatible with the metric on that space, for example, the Φ on $\mathrm{Im}\,\widetilde{\mathbb{O}} \cong \mathbb{R}^7$ with the metric $\langle \cdot, \cdot \rangle$ defined in Subsection 1.4.1, for an appropriate isomorphism. Since the G_2-structure is parallel, ϕ_x extends to a parallel local 3-form ϕ on M such that fiber of the G_2-structure over any point y again comprises the frames with respect to which ϕ_y has the coordinate representation of the fixed 3-form on $\mathrm{Im}\,\widetilde{\mathbb{O}}$. Patching these local 3-forms together gives a global parallel 3-form of split type, and it is compatible with the metric since Φ is compatible with $\langle \cdot, \cdot \rangle$.

(⇐) If (N,h) admits a parallel 3-form of split type compatible with the metric in the above sense, its holonomy is contained in the stabilizer in $\mathrm{GL}(7,\mathbb{R})$ of a 3-form of split type on $\mathbb{R}^7$, namely, G_2. □

We can realize G_2 in an alternate way, namely as the stabilizer in $\mathrm{Spin}^+(3,4)$ (the connected component of the identity of $\mathrm{Spin}(3,4)$) of a nonisotropic (that is, non-null) spinor $\psi \in \Delta_{3,4} \cong \mathbb{R}^8$ [Kat98, Corollary 2.1]. This gives the following:

Proposition 1.6.9. *If a pseudo-Riemannian manifold (N,h) admits a nonisotropic spinor ψ, that is a section of the spin bundle associated to $\Delta_{3,4}$, parallel with respect to the induced spin connection, then* $\mathrm{Hol}(N,h)$ *is contained in* G_2*.*

We can canonically assign a 3-form in $\Lambda^3(\mathbb{R}^{3,4})^*$ to any nonisotropic spinor $\psi \in \Delta_{3,4}$: For such a spinor, the map $\mathbb{R}^{3,4} \mapsto \{\varphi\}^\perp \subset \Delta_{3,4}$ defined by $X \mapsto X \cdot \psi$ is an isomorphism (if ψ is null instead, the map has nullity 3), and so passing to bundles we may define a bundle map $\times : TN \otimes TN \to TN$ (that is, $(2,1)$-tensor) by

$$YX\psi + h(X,Y)\psi = (X \times Y)\psi.$$

Checking shows that that lowering an index of $\times$ gives a 3-form Φ of split type [Kat98]. By construction, if ψ is parallel with respect to the induced spin connection, Φ is parallel with respect to ∇^h.

1.6.2 *Normal conformal and ambient holonomy*

We may likewise define the holonomy for a general connection $\nabla^E : \Gamma(E) \to \Gamma(E \otimes T^*N)$ on a vector bundle $E \to N$, which induces parallel transport maps P_γ analogous to the one defined above. For any curve $\gamma : [0,1] \to N$, a local section s of E is parallel along γ if $\nabla_{\gamma'(t)} s = 0$. This equation is a first-order, linear ordinary differential equation, so define the parallel transport map $P_\gamma : E_{\gamma(0)} \to E_{\gamma(1)}$ to be (linear) map that sends $X \in E_{\gamma(0)}$ to $s(\gamma(1))$, where s is the unique solution to $\nabla_{\gamma'(t)} s = 0$ satisfying $s(\gamma(0)) = X$. By the same arguments in Subsection 1.6.1, if γ is a loop based at $x \in N$, then P_γ is in $\mathrm{GL}(E_x)$.

Definition 1.6.10. The **holonomy (group)** of the connection ∇^E is the group

$$\mathrm{Hol}_x(\nabla^E) := \{P_\gamma : \gamma \in \Omega_x(M)\} \leq \mathrm{GL}(E_x),$$

Specializing E to TN and ∇^E to the Levi-Civita connection ∇^h of a metric h on N recovers the notion of holonomy defined in Subsection 1.6.1.

Taking E to be the tractor bundle $\mathcal{T}$ defines a natural notion of holonomy for conformal structures.

Definition 1.6.11. The **normal conformal holonomy (group)** of the conformal structure (M, c) is the holonomy $\mathrm{Hol}(\nabla^{\mathcal{T}})$ of the standard tractor connection.

One can show that this agrees with the standard definition of this group: Let $(E \to M, \omega)$ be the parabolic geometry of type $(\mathfrak{o}(p+1, q+1), \dot{P})$. Then, the normal conformal holonomy is just the holonomy of the principal connection on the $\mathrm{O}(p+1, q+1)$-principal bundle $E \times_{\dot{P}} \mathrm{O}(p+1, q+1) \to M$ defined by extending ω by equivariance.

By construction, the tractor metric $g^{\mathcal{T}}$ of a conformal structure (M, c) of signature, say, (p, q), has signature $(p+1, q+1)$ and is $\nabla^{\mathcal{T}}$-parallel, so $\nabla^{\mathcal{T}}$ preserves lengths, and hence $\mathrm{Hol}(\nabla^{\mathcal{T}}) \leq \mathrm{O}(p+1, q+1)$.

Since the tractor connection of a conformal structure (M, c) is essentially given by restricting the connection $\widetilde{\nabla}$ of any ambient metric $(\widetilde{M}, \widetilde{g})$ of c to $\mathcal{G} \subset \widetilde{M}$, and any loop in $\mathcal{G}$ is also a loop in $\widetilde{M}$, we always have $\mathrm{Hol}(\nabla^{\mathcal{T}}) \leq \mathrm{Hol}(\widetilde{\nabla})$.

The following equality result for Einstein conformal structures can be found in [Lei04] (for Einstein constant $\lambda \neq 0$) and [Lei05] ($\lambda = 0$).

Proposition 1.6.12. *If c is conformally Einstein,* $\operatorname{Hol}(\nabla^{\mathcal{T}}) = \operatorname{Hol}(\widetilde{\nabla}^{\mathrm{can}})$*, where* $\widetilde{\nabla}^{\mathrm{can}}$ *is the Levi-Civita connection of the canonical ambient metric* $\widetilde{g}^{\mathrm{can}}$ (1.21) *of any Einstein representative of c.*

1.6.3 Leistner and Nurowski's examples

We may associate to any generic 2-plane field $\mathbb{D}$ on a 5-manifold the normal conformal holonomy $\operatorname{Hol}(c_{\mathbb{D}})$, which again we may identify with its tractor holonomy $\operatorname{Hol}(\nabla^{\mathcal{T}})$. Recall that if $\mathbb{D}$ is real-analytic, then so is $c_{\mathbb{D}}$, and thus by the discussion in Subsection 1.6.2 we may investigate the holonomy $\operatorname{Hol}(\widetilde{g}_{\mathbb{D}})$ of a real-analytic ambient metric $\widetilde{g}_{\mathbb{D}}$ of $c_{\mathbb{D}}$. In principle, the holonomy may depend on the choice of metric $\widetilde{g}_{\mathbb{D}}$, but we will always be able to restrict the metric to some homogeneous domain so that the relevant results apply. Recall from Subsection 1.5.2 that any function $F(x, y, p, q, z)$ for which F_{qq} is nonvanishing determines a generic 2-plane field $\mathbb{D}_F$, and that any plane field $\mathbb{D}$ can be locally realized as a plane field $\mathbb{D}_F$ for some such F. Then, if F is real-analytic, by the observation in 1.5.3 so is $c_{\mathbb{D}_F}$, so the above yields a construction $F \rightsquigarrow \operatorname{Hol}(\widetilde{g}_{\mathbb{D}_F})$. Summarizing, the chain of constructions is

$$F \mapsto \mathbb{D}_F \mapsto c_F \mapsto \widetilde{g}_F \mapsto \operatorname{Hol}(\widetilde{g}_F),$$

where we denote $c_F := c_{\mathbb{D}_F}$ and $\widetilde{g}_F := \widetilde{g}_{\mathbb{D}_F}$.

Strikingly, Leistner and Nurowski exhibited an eight-parameter class of (real-analytic) functions F for which the ambient holonomy of $\widetilde{g}_F$ is equal to G_2. This result in large part motivated the investigation of what became the primary application in this work, namely, the relationship between general real-analytic local fields $\mathbb{D}$ and G_2 holonomy.

Example 1.6.13. [LN10, Theorem 1] Define for $\mathbf{a} = (a_0, \ldots, a_6) \in \mathbb{R}^7$ and $b \in \mathbb{R}$ the function $F[\mathbf{a}, b]$ by

$$F[\mathbf{a}, b](x, y, p, q, z) = q^2 + a_0 + a_1 p + a_2 p^2 + a_3 p^3 + a_4 p^4 + a_5 p^5 + a_6 p^6 + bz.$$

If at least one of a_3, a_4, a_5, or a_6 is nonzero, then, $\operatorname{Hol}(\widetilde{g}_{F[\mathbf{a},b]}) = \mathrm{G}_2$.

Remarkably, for any $\mathbf{a}$ and b, the conformal class $c_{F[\mathbf{a},b]}$ admits a representative $\widehat{g}$ such that in normal form with respect to that representative the real-analytic ambient metric

$\widetilde{g}_{F[\mathbf{a},b]}$ is given a by second-order polynomial in ρ:

$$\widetilde{g}_{F[\mathbf{a},b]} := 2\rho\, dt^2 + 2t\, dt\, d\rho + t^2(\widehat{g} + 2P\rho + t^2 B\rho^2),$$

where P and B respectively are the Schouten and Bach tensors. As usual, we suppress pullback notation.

We outline here Leistner and Nurowski's proof that $\mathrm{Hol}(\widetilde{g}_{F[\mathbf{a},b]}) = \mathrm{G}_2$ for the values of $(\mathbf{a}, b)$ given in 1.6.13. First, they construct an explicit nonisotropic spinor on the ambient space parallel with respect to the spin connection induced by the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{g}_{F[\mathbf{a},b]}$, and so Proposition 1.6.9 shows that the holonomy of each such metric is contained in G_2. To show that the holonomy is equal to G_2, they eliminate the other possibilities.

Proposition 1.6.14. *Suppose a simply-connected, signature-$(3,4)$ pseudo-Riemannian manifold (N, h) has holonomy contained in G_2. Then, at least one of the following holds:*

- *h is locally symmetric, that is, $\nabla^h R = 0$*
- *h admits a parallel line field in a neighborhood of every point*
- *h admits a parallel, totally null 2-plane field*
- $\mathrm{Hol}(h) = \mathrm{G}_2$

Proof. Fix an arbitrary point $x \in N$. If $\mathrm{Hol}(h)$ acts irreducibly on T_xN, then by Berger's List (Theorem 1.6.4), either $\mathrm{Hol}(h) = \mathrm{G}_2$ or h is locally symmetric. So henceforth suppose $\mathrm{Hol}(h)$ acts reducibly, say, that it preserves the proper subspace $V \subset T_xM$. Since $\dim T_xN = 7$, by replacing V with $V^\perp$ if necessary, we may assume $\dim V \leq 3$.

First assume that V is nondegenerate, so that $V \cap V^\perp = \{0\}$. Then, by the de Rham theorem, N can be factored in some neighborhood of x as a nontrivial product $(N_1, h_1) \times (N_2, h_2)$, say, with $\dim N_1 \leq 3$. By Bonan's Theorem (Theorem 1.6.7), h is Ricci-flat, and thus so are h_1 and h_2. Since $\dim N_1 \leq 3$, h_1 is actually flat and thus admits a parallel vector field. Hence, its span is a parallel local line field on N near x.

Now assume that V is degenerate. If $\dim V = 1$ or $\dim V = 2$, then V itself is a parallel null line or 2-plane field, respectively. Now suppose $\dim V = 3$; in this case [Kat98, Kop97] give that there is a line of pure null spinors parallel with respect to the spin connection induced by h. Let Φ denote a parallel 3-form stabilized by G_2 compatible with the metric h in the sense that, up to a constant factor,

$$(\cdot \lrcorner \Phi) \wedge (\cdot \lrcorner \Phi) \wedge \lrcorner \Phi = h \operatorname{vol}_h .$$

Let ψ be the spinor field that corresponds to Φ. Then, for any nonvanishing local section ζ of the line of null spinors, define the vector field V by transposing Clifford multiplication, that is, by

$$h(V, Y) = \langle Y \cdot \psi, \zeta \rangle ,$$

where $\langle \cdot, \cdot \rangle$ is the metric on the spin bundle. By construction, scaling ζ scales V, so this transposition defines a parallel local line field. (In fact, these local fields agree on their overlap, defining a parallel global line field.) □

To eliminate the first three possibilities for the metrics indicated in Example 1.6.13, Leistner and Nurowski [LN10] proved the following technical result (which combines the statements of several results in that article).

Lemma 1.6.15. *Let $\mathbb{D}$ be a real-analytic, oriented, generic 2-plane field on a 5-manifold M, and let $(\widetilde{M}, \widetilde{g}_{\mathbb{D}})$ be the real-analytic ambient manifold of the induced conformal structure $c_{\mathbb{D}}$. Then,*

- *If $\widetilde{g}_{\mathbb{D}}$ admits a parallel line field, then there is a dense subset $U \subseteq M$ such that $c_{\mathbb{D}}|_U$ is locally Einstein.*

- *If $\widetilde{g}_{\mathbb{D}}$ admits a parallel, totally null 2-plane field, then for any point there is a representative $g \in c_{\mathbb{D}}$ for which there is a local g-parallel null line field $L \subset TM$ near that point such that $\operatorname{Ric}_g(L^\perp, \cdot) = 0$; in this case, for any $x \in M$, $W(Y, K, K, X) = 0$ for $K \in L_x$, $Y \in T_x M$, and $X \in L_x^\perp$).*

By this lemma and Proposition 1.6.14, to show that the real-analytic ambient metric $\widetilde{g}_{\mathbb{D}}$ of a real-analytic, oriented, generic 2-field field $\mathbb{D}$ on a 5-manifold M has holonomy equal to G_2 (and not a proper subgroup thereof), it is enough to show that (1) there is some nonempty open subset of M on which $c_{\mathbb{D}}$ is not conformally Einstein, and (2) there is some nonempty open set such that that the manifold does not admit a pair (g, L) as in the lemma.

To do this for the indicated plane fields $\mathbb{D}_{F[\mathbf{a},b]}$, Leistner and Nurowski explicitly compute tensorial data associated to the representative metrics of $c_{\mathbb{D}_{F[\mathbf{a},b]}}$ given by (A.1) (some of it is recorded here in Appendix A.2) and show that neither condition in Lemma 1.6.15 can be satisfied, completing the proof.

Chapter 2

PARALLEL TRACTOR EXTENSION

2.1 The Parallel Tractor Extension Theorem

Section 1.3 defined the standard tractor bundle $\mathcal{T}$ of a conformal structure (M, c) as the restriction $T\widetilde{M}|_{\mathcal{G}}$ of the ambient tangent bundle to the metric bundle, modulo the restriction of the natural dilation action. Moreover, it showed that via this construction, the restriction of the ambient connection, $\widetilde{\nabla}$, to the metric bundle descends via the dilation action to a connection $\nabla^{\mathcal{T}}$ on $\mathcal{T}$. So by construction, any $\nabla^{\mathcal{T}}$-parallel section of $\mathcal{T}$ can be identified with a section of $T\widetilde{M}|_{\mathcal{G}}$ parallel with respect to the restriction of $\widetilde{\nabla}$ (in particular, with derivatives taken tangent to $\mathcal{G}$). We investigate the conditions under which a parallel tractor can be extended to a suitable parallel vector field on the ambient manifold, or more precisely, under which there is a parallel vector field on the ambient manifold (or at least an open subset of it containing the metric bundle) whose restriction to $\mathcal{G}$ is the parallel tractor identified as above. We then study the same questions for parallel tractor tensors. By lowering indices using the tractor and ambient metrics, for convenience we can derive results just for covariant tractor tensors; one can then raise indices as desired to achieve the corresponding results for contravariant and mixed tractor tensors.

Throughout this section, (M, c) is a conformal manifold of dimension $n > 2$. Denote the metric bundle of c by $\mathcal{G}$, the standard tractor bundle by $\mathcal{T}$, its connection by $\nabla^{\mathcal{T}}$, let $(\widetilde{M}, \widetilde{g})$ be an ambient manifold for c, and let $\widetilde{\nabla}$ be the Levi-Civita connection of $\widetilde{g}$.

2.1.1 The main theorem

Recall from section 1.3 that the covariant rank-r tractor tensors can be identified with the restricted sections $\chi \in \Gamma(\bigotimes^r T^*\widetilde{M}|_{\mathcal{G}})$ satisfying the homogeneity condition $(\delta^s)^*\chi = s^r\chi$. That development also showed that the tractor connection can be realized in terms of the ambient connection by the formula

$$\nabla^{\mathcal{T}}_v \chi = \widetilde{\nabla}_V \chi,$$

for any $V \in (T\pi)^{-1}(v)$. (Recall that the right-hand side of the above formula does not depend on the ambiguity of this choice, because χ has the stated homogeneity, which is equivalent to the identity $\widetilde{\nabla}_T \chi \equiv 0$.) We will sometimes invoke this identification without comment.

By the above realization of $\nabla^{\mathcal{T}}$ in terms of $\widetilde{\nabla}$, the restriction of any parallel ambient tensor $\widetilde{\chi} \in \Gamma(\bigotimes^r T^*\widetilde{M})$ to $\mathcal{G}$ is a parallel tractor tensor. Since we are interested in reversing this restriction, and any parallel ambient tensor $\widetilde{\chi}$ in particular satisfies $\widetilde{\nabla}_T \widetilde{\chi} \equiv 0$, or equivalently, $(\delta^s)^* \widetilde{\chi} = s^r \widetilde{\chi}$, we define the following:

Definition 2.1.1. An **ambient extension** of a rank-r tractor tensor χ is a section $\widetilde{\chi} \in \Gamma(\bigotimes^r T^*U)$ satisfying $\widetilde{\chi}|_{\mathcal{G}} = \chi$ and $(\delta^s)^* \widetilde{\chi} = s^r \widetilde{\chi}$, where U is some open, dilation-invariant neighborhood of $\mathcal{G}$ in $\widetilde{M}$.

The following theorem, the first main novel result in this dissertation, states that there are always ambient extensions parallel in at least a limited sense.

Theorem 2.1.2 (Parallel Tractor Extension Theorem)**.** *Let (M, c) be a conformal manifold of dimension $n > 2$, and let $\widetilde{g}$ be an ambient metric for c.*

- *If n is odd, any parallel tractor tensor χ admits an ambient extension $\widetilde{\chi}$ satisfying $\widetilde{\nabla} \widetilde{\chi} = O(\rho^\infty)$.*

- *If n is even, any parallel tractor tensor χ admits an ambient extension $\widetilde{\chi}$ satisfying $\widetilde{\nabla} \widetilde{\chi} = O(\rho^{n/2-1})$.*

(Here, ρ denotes the ambient coordinate determined by an arbitrary representative metric $g \in c$.)

Gover proved this theorem in the special case when n is odd and $r = 1$ using a method fundamentally different from that in the below proof; the same argument also applies to the case n is even and $r = 1$.

In the case that c is real-analytic and $\widetilde{g}$ is the unique real-analytic ambient metric (up to extension and diffeomorphism fixing $\mathcal{G}$ pointwise), this theorem specializes to the following, in particular giving in odd dimensions that a parallel tractor tensor always extends to a bona fide parallel ambient tensor; this is the fact used in the main application of this theorem in this dissertation, in Section 3.1.

Theorem 2.1.3 (Parallel Tractor Extension Theorem: odd, real-analytic case)**.** *Let (M, c) be a real-analytic conformal manifold of odd dimension, and let $(\widetilde{M}, \widetilde{g})$ be a real-analytic ambient manifold for c. Any parallel tractor tensor χ on M admits a bona-fide (and real-analytic) parallel ambient extension.*

Before we give the proof of Theorem 2.1.2, we collect some technical facts about the curvature of ambient metrics that will be used there, and prove two additional technical lemmas.

Recall that a choice of representative $g \in c$ induces a trivialization $\mathcal{G} \leftrightarrow M \times \mathbb{R}^+$ and thus defines an embedding $M \hookrightarrow \mathcal{G}$ by $x \mapsto (1, x)$ and, composing with the embedding $\mathcal{G} \to \widetilde{M}$, an embedding $M \hookrightarrow \widetilde{M}$ (we will sometimes identify M with its images under these embeddings). Via the normal form of the ambient metric, the choice g determines an identification of $\widetilde{M}$ with an open subset of $\mathbb{R}^+ \times M \times \mathbb{R}$ containing $\mathbb{R}^+ \times M$, and this induces a splitting of indices, $A \leftrightarrow (0, a, \infty)$; explicitly, $\partial_0 = \partial_t$ and $\partial_\infty = \partial_\rho$.

Lemma 2.1.4. *For an ambient metric in normal form with respect to $g \in c$, on $M \subset \widetilde{M}$ the curvature $\widetilde{R}$ satisfies*

$$(2s+1)\widetilde{R}_{ABC\infty,\underbrace{\infty\cdots\infty}_{s-1}} = \widetilde{R}_{ABC}{}^{d}{}_{,d\underbrace{\infty\cdots\infty}_{s-1}}$$

provided either n is odd, or n is even and $||ABC|| \leq n - 2s + 1$, where $||\cdot||$ denotes the strength of the argument multi-index.

Proof. All expressions are evaluated on M (that is, at $t = 1$, $\rho = 0$). First, assume that n is odd. Contracting the Second Bianchi Identity reads

$$2\widetilde{g}^{DE}\widetilde{R}_{CD[AB,E]\underbrace{\infty\cdots\infty}_{s-1}} = 0$$

Since $\widetilde{g}$ is Ricci-flat to infinite order, this reduces to

$$\widetilde{g}^{DE}\widetilde{R}_{ABCD,E\underbrace{\infty\cdots\infty}_{s-1}} = 0$$

after applying a symmetry. Expanding the trace using the normal form of $\widetilde{g}$ at $t = 1$, $\rho = 0$ gives

$$\widetilde{R}_{ABC0,\infty\underbrace{\infty\cdots\infty}_{s-1}} + g^{de}\widetilde{R}_{ABCd,e\underbrace{\infty\cdots\infty}_{s-1}} + \widetilde{R}_{ABC\infty,0\underbrace{\infty\cdots\infty}_{s-1}} = 0.$$

Now, Proposition 1.2.13 gives

$$\widetilde{R}_{ABC0,\underbrace{\infty\cdots\infty}_{s}} = -s\widetilde{R}_{ABC\infty,\underbrace{\infty\cdots\infty}_{s-1}}, \qquad \widetilde{R}_{ABC\infty,0\underbrace{\infty\cdots\infty}_{s-1}} = -(s+1)\widetilde{R}_{ABC\infty,\underbrace{\infty\cdots\infty}_{s-1}}.$$

Substituting and rearranging yield the identity for the case n is odd. If n is even, the condition on the strength of ABC guarantees that the relevant component of the derivative of the Ricci curvature of $\widetilde{g}$ vanishes, so that the Bianchi identity reduces as in the above proof. □

Notice that when n is even, in the special case that c is obstruction-flat and $\widetilde{g}$ is an infinite-order ambient metric, the strength condition can be eliminated.

Herein, $\mathbf{A}$ denotes the arbitrary multi-index $A_1 \cdots A_r$.

We invoke the following lemma twice in the proof of the theorem.

Lemma 2.1.5. *Suppose c is a conformal structure with metric bundle $\mathcal{G}$ and ambient manifold $(\widetilde{M}, \widetilde{g})$, let $\widetilde{\nabla}$ denote the Levi-Civita connection of $\widetilde{g}$, and suppose that $\beta \in \Gamma(\bigotimes^r T^*\widetilde{M})$ satisfies $(\widetilde{\nabla}^l\beta)|_{\mathcal{G}} = 0$ for $l \in \{1, \ldots, s-1\}$, for some $s > 0$. Then, the value of a component $\beta_{\mathbf{A},B_0\cdots B_s}|_{\mathcal{G}}$ of $(\widetilde{\nabla}^{s+1}\beta)|_{\mathcal{G}}$ does not depend on the order of the last s indices, $B_1, \ldots, B_s$.*

Proof. Consider any v, $1 \le v \le s-1$; we show that transposing the indices B_v and B_{v+1} preserves the value of the component $\beta_{\mathbf{A},B_0\cdots B_s}$ on $\mathcal{G}$. This will prove the claim, because such transpositions generate all of the permutations of $B_1, \ldots, B_s$. (Possibly) differentiating the Ricci identity gives

$$\beta_{\mathbf{A},B_0\cdots B_{v-1}B_{v+1}B_vB_{v+2}\cdots B_s}$$
$$= \Bigg(\beta_{\mathbf{A},B_0\cdots B_{v+1}} - \sum_{u=1}^{r} \widetilde{R}_{B_vB_{v+1}}{}^{Q}{}_{A_u}\beta_{A_1\cdots A_{u-1}QA_{u+1}\cdots A_r,B_0\cdots B_{v-1}}$$
$$- \sum_{u=1}^{v-1} \widetilde{R}_{B_vB_{v+1}}{}^{Q}{}_{B_u}\beta_{\mathbf{A},B_0\cdots B_{u-1}QB_{u+1}\cdots B_{v-1}}\Bigg)_{B_{v+2}\cdots B_s}.$$

Distributing the derivatives and iteratively applying the Leibniz rule over the sums, the above becomes

$$\beta_{\mathbf{A},B_0\cdots B_{v-1}B_{v+1}B_vB_{v+2}\cdots B_s} = \beta_{\mathbf{A},B_0\cdots B_{s+1}} + \gamma_{\mathbf{A}B_0\cdots B_s},$$

where γ is some $\mathbb{Z}$-linear combination of covariant derivatives of $\widetilde{R}$ contracted into covariant derivatives of β of order between 1 and $s-1$ inclusively. Restricting to $\mathcal{G}$, those latter derivatives vanish by hypothesis, leaving the desired equality. □

Proof of Parallel Tractor Extension Theorem. First, if $r = 0$, so that χ is just a constant function f, then the constant function on $\widetilde{M}$ taking on the same value is a parallel extension of f (and the only such extension).

So, henceforth assume $r \geq 1$. Fix a representative metric $g \in c$, and put $\widetilde{g}$ into normal form with respect to g. Define $\widetilde{\chi} \in \Gamma(\bigotimes^r T^*\widetilde{M})$ by parallel transport of χ in the ρ direction: Explicitly, for each $(t_0, x_0, \rho_0) \in \widetilde{M}$, define $\widetilde{\chi}_{(t_0,x_0,\rho_0)}$ by parallel transport of $\chi_{(t_0,x_0,0)}$ along the curve $\rho \mapsto (t_0, x_0, \rho)$. Since parallel translation commutes with dilations, $\widetilde{\chi}$ is an ambient extension. In fact, any parallel ambient extension of χ must be parallel along these curves, so if χ admits any parallel ambient extension, it must be the candidate $\widetilde{\chi}$. In the language of differential equations, the condition that an extension of χ be parallel is an overdetermined system of partial differential equations with initial value condition, and $\widetilde{\chi}$ is the unique solution to the ordinary differential system in ρ comprising a subset of those equations and the same initial value condition.

We show below that $\widetilde{\nabla}^s\widetilde{\chi}$ vanishes along $\mathcal{G}$ (that is, where $\rho = 0$) for all $s \geq 1$ if n is odd and for $1 \leq s \leq \frac{n}{2} - 1$ if n is even. Then, iteratively both expanding all but the first innermost covariant derivative in the Christoffel symbols (1.22) and substituting shows that $\widetilde{\nabla}\widetilde{\chi}$ vanishes to the stated order, completing the proof.

We first show that $\widetilde{\chi}_{\mathbf{A},\underbrace{\infty\cdots\infty}_{s}} = 0$ (identically) on $\widetilde{M}$. The case $s = 1$ holds by definition: $\widetilde{\chi}$ is parallel in the ρ direction. Now, inductively suppose that $\widetilde{\chi}_{\mathbf{A},\underbrace{\infty\cdots\infty}_{l}} = 0$ on $\widetilde{M}$ for $1 \leq l \leq s$; expanding in Christoffel symbols gives

$$\widetilde{\chi}_{\mathbf{A},\underbrace{\infty\cdots\infty}_{s+1}} = \partial_\infty \widetilde{\chi}_{\mathbf{A},\underbrace{\infty\cdots\infty}_{s}} - \sum_{u=1}^{r} \widetilde{\Gamma}^{Q}_{\infty A_u} \widetilde{\chi}_{A_1\cdots A_{u-1}QA_{u+1}\cdots A_r,\underbrace{\infty\cdots\infty}_{s}} - \sum_{u=1}^{s} \widetilde{\Gamma}^{Q}_{\infty\infty} \widetilde{\chi}_{\mathbf{A},\underbrace{\infty\cdots\infty}_{u-1}Q\underbrace{\infty\cdots\infty}_{s-u}}$$

The first and second terms vanish identically by the inductive hypothesis. Consulting (1.22) shows that $\widetilde{\Gamma}^{Q}_{\infty\infty} = 0$, and so the last term vanishes, too, proving the inductive claim.

We proceed to the main induction, showing now for all $s \geq 1$ (and, if n is even, $s \leq \frac{n}{2} - 1$) that all the components $\widetilde{\chi}_{\mathbf{A},B_0\cdots B_{s-1}}$ of $\widetilde{\nabla}^s\widetilde{\chi}$ vanish on $\mathcal{G}$.

In the base case, $s = 1$, the statement that $\widetilde{\chi}_{\mathbf{A},B_0} = 0$ for $B_0 = 0, b_0$ on $\mathcal{G}$ is equivalent to the hypothesis that χ is a parallel tractor; as in the base case of the previous induction, the identity $\widetilde{\chi}_{\mathbf{A},\infty} = 0$ follows from the definition of $\widetilde{\chi}$, completing the base case.

Now, assume inductively that $\widetilde{\nabla}^l\widetilde{\chi} = 0$ on $\mathcal{G}$ for $1 \leq l \leq s$; if n is even also assume $s < \frac{n}{2} - 1$. To show that $\widetilde{\chi}_{\mathbf{A},B_0\cdots B_s} = 0$ and so complete the induction step, we partition the step into several cases characterized by the specializations of the multi-index $B_0 \cdots B_s$.

Case A ($B_s \neq \infty$)

Expanding in Christoffel symbols gives

$$\widetilde{\chi}_{\mathbf{A},B_0\cdots B_s} = \partial_{B_s}\widetilde{\chi}_{\mathbf{A},B_0\cdots B_{s-1}} + \gamma,$$

where γ is a $C^\infty(\widetilde{M})$-linear combination of components of $\widetilde{\nabla}^s\widetilde{\chi}$, which by the inductive hypothesis vanishes on $\mathcal{G}$. Since $B_s \neq \infty$, ∂_{B_s} is tangent to $\mathcal{G}$, and so the first term on the right in the above expression also vanishes by the inductive hypothesis.

Case B ($B_s = \infty$, $B_l \neq \infty$ for some $l > 0$)

By Lemma 2.1.5, exchanging B_l and B_s preserves the value of the component and so reduces this case to Case A.

Case C ($B_1 = \cdots = B_s = \infty$)

Denote $B = B_0$. The induction argument preceding this main induction completes the argument when $B = \infty$. So, henceforth assume that $B \neq \infty$. By the homogeneity of the derivatives of $\widetilde{\chi}$, it suffices to prove the claim on $M \subset \mathcal{G}$ (that is, where $t = 1$, $\rho = 0$); all of the following expressions are implicitly evaluated there.

Differentiating the Ricci identity gives

$$\widetilde{\chi}_{\mathbf{A},B\underbrace{\infty\cdots\infty}_{s}} = \widetilde{\chi}_{\mathbf{A},\infty B\underbrace{\infty\cdots\infty}_{s-1}} + \left(\sum_{u=1}^{r} \widetilde{R}^{Q}{}_{A_u B\infty}\widetilde{\chi}_{A_1\cdots A_{u-1}QA_{u+1}\cdots A_r}\right)_{,\underbrace{\infty\cdots\infty}_{s-1}}.$$

By Case B, the first term on the right-hand side is zero. Now, iteratively apply the Leibniz rule to the second term: All of the terms in which at least one derivative index is applied to $\widetilde{\chi}$ are zero by the inductive hypothesis, leaving just the term in which all derivative indices are applied to the curvature factor:

$$\widetilde{\chi}_{\mathbf{A},B\underbrace{\infty\cdots\infty}_{s}} = \sum_{u=1}^{r} \widetilde{R}^{Q}{}_{A_u B\infty,\underbrace{\infty\cdots\infty}_{s-1}}\widetilde{\chi}_{A_1\cdots A_{u-1}QA_{u+1}\cdots A_r}.$$

If $B = 0$, then by the first equation in Proposition 1.2.13, the right-hand side vanishes, so we henceforth assume $B = b$.

We now invoke Lemma 2.1.4. Let L be the index corresponding to Q when it is lowered; then, when n is even, $s \leq \frac{n}{2} - 2$ and $n \geq 4$, and so $||LA_u b|| \leq 5 \leq n - 2s + 1$. Thus, the hypotheses of the lemma always hold. Applying it to the previous display equation gives

$$\begin{aligned}(2s+1)\widetilde{\chi}_{\mathbf{A},b\underbrace{\infty\cdots\infty}_{s}} &= (2s+1)\sum_{u=1}^{r} \widetilde{R}^{Q}{}_{A_u b\infty,\underbrace{\infty\cdots\infty}_{s-1}}\widetilde{\chi}_{A_1\cdots A_{u-1}QA_{u+1}\cdots A_r} \\ &= \sum_{u=1}^{r} \widetilde{R}^{Q}{}_{A_u b}{}^{d}{}_{,d\underbrace{\infty\cdots\infty}_{s-1}}\widetilde{\chi}_{A_1\cdots A_{u-1}QA_{u+1}\cdots A_r}.\end{aligned}$$

Reversing the previous argument involving the Leibniz rule and then applying the Ricci Identity gives

$$\begin{aligned}(2s+1)\widetilde{\chi}_{\mathbf{A},b\underbrace{\infty\cdots\infty}_{s}} &= \left(\sum_{u=1}^{r} \widetilde{R}^{Q}{}_{A_u b}{}^{d}\widetilde{\chi}_{A_1\cdots A_{u-1}QA_{u+1}\cdots A_r}\right)_{,d\underbrace{\infty\cdots\infty}_{s-1}} \\ &= (\widetilde{\chi}_{\mathbf{A},b}{}^{d} - \widetilde{\chi}_{\mathbf{A},}{}^{d}{}_{b})_{d\underbrace{\infty\cdots\infty}_{s-1}} \\ &= \widetilde{\chi}_{\mathbf{A},b}{}^{d}{}_{d\underbrace{\infty\cdots\infty}_{s-1}} - \widetilde{\chi}_{\mathbf{A},}{}^{d}{}_{bd\underbrace{\infty\cdots\infty}_{s-1}}.\end{aligned}$$

By Lemma 2.1.5, we may permute indices to give

$$(2s+1)\widetilde{\chi}_{\mathbf{A},b\underbrace{\infty\cdots\infty}_{s}} = \widetilde{\chi}_{\mathbf{A},b}{}^{d}{}_{\underbrace{\infty\cdots\infty}_{s-1}d} - \widetilde{\chi}_{\mathbf{A},}{}^{d}{}_{b\underbrace{\infty\cdots\infty}_{s-1}d}.$$

Now, expand the final covariant differentiation of each term on the right-hand size in terms of Christoffel symbols. The previous parts of the induction step show that all the components of $\widetilde{\nabla}^{s+1}\widetilde{\chi}$ are 0 on $\mathcal{G}$ except perhaps $\widetilde{\chi}_{\mathbf{A},b\underbrace{\infty\cdots\infty}_{s}}$; using this and consulting (1.22) yields

$$\widetilde{\chi}_{\mathbf{A},b}{}^{d}{}_{\underbrace{\infty\cdots\infty}_{s-1}d} = \partial_d\widetilde{\chi}_{\mathbf{A},b}{}^{d}{}_{\underbrace{\infty\cdots\infty}_{s-1}} + \widetilde{\Gamma}^{d}_{d0}\widetilde{\chi}_{\mathbf{A},b}{}^{0}{}_{\underbrace{\infty\cdots\infty}_{s-1}} = n\widetilde{\chi}_{\mathbf{A},b\underbrace{\infty\cdots\infty}_{s}}$$

$$\widetilde{\chi}_{\mathbf{A},}{}^{d}{}_{b\underbrace{\infty\cdots\infty}_{s-1}d} = \partial_d\widetilde{\chi}_{\mathbf{A},}{}^{d}{}_{b\underbrace{\infty\cdots\infty}_{s-1}} - \widetilde{\Gamma}^{\infty}_{db}\widetilde{\chi}_{\mathbf{A},}{}^{d}{}_{\underbrace{\infty\cdots\infty}_{s}} = \widetilde{\chi}_{\mathbf{A},b\underbrace{\infty\cdots\infty}_{s}}.$$

Substituting these two equations into the previous display equation and rearranging then gives

$$(2s+2-n)\widetilde{\chi}_{\mathbf{A},b\underbrace{\infty\cdots\infty}_{s}} = 0.$$

Since $s \neq \frac{n}{2} - 1$ by hypothesis, dividing gives that the component $\widetilde{\chi}_{\mathbf{A},b\underbrace{\infty\cdots\infty}_{s}}$ vanishes on $\mathcal{G}$ as desired, completing the induction. □

We give a few examples of explicit extensions of parallel tractors.

Example 2.1.6. (Constants) As observed at the beginning of the proof, any parallel 0-tractor (constant function) trivially admits a unique ambient extension to any ambient manifold $(\widetilde{M}, \widetilde{g})$, namely, the constant function on $\widetilde{M}$ that takes on the same value.

Example 2.1.7. (Tractor metric) Again trivially, for any choice of ambient metric $\widetilde{g}$ of c, $\widetilde{g}$ is itself a parallel ambient extension of the tractor metric $g^{\mathcal{T}}$. Likewise, the identity endomorphism $\mathrm{id}_{T\widetilde{M}}$ of the ambient tangent bundle is a parallel ambient extension of the identity endomorphism of the tractor bundle.

Example 2.1.8. (Einstein conformal scale) Suppose c has an Einstein representative g, with, say, $\operatorname{Ric} g = 2(n-1)\lambda g$. Recall from Subsection 1.3.3 that with respect to g, the Einstein scale 1 corresponds to the parallel cotractor $\xi = (1, 0, -\lambda)$, and from Subsection 1.2.5 that c admits a canonical, Ricci-flat ambient metric, namely,

$$\widetilde{g}^{\mathrm{can}}(t, x, \rho) = 2\rho\, dt^2 + 2t\, dt\, d\rho + t^2(1+\lambda\rho)^2 g(x).$$

Checking directly shows that $\widetilde{\xi} = (1-\lambda\rho)\, dt - t\lambda\, d\rho$ is an ambient extension of ξ parallel with respect to the Levi-Civita connection of $\widetilde{g}^{\mathrm{can}}$.

The following proposition shows that on an Einstein conformal structure, parallel tractor tensors (of arbitrary rank) always admit an ambient extension parallel with respect to the canonical ambient metric. This is roughly equivalent to to [Lei04, Proposition 2] for non–Ricci-flat structures and to [Lei05, Theorem 4.2] for Ricci-flat structures.

Proposition 2.1.9. *Suppose c has an Einstein representative g, with, say,* $\operatorname{Ric} g = 2\lambda(n-1)g$. *Then, any parallel tractor tensor χ admits an ambient extension parallel with respect to the Levi-Civita connection of the canonical ambient metric $\widetilde{g}^{\mathrm{can}}$.*

To prove this efficiently, we use the following lemma.

Lemma 2.1.10. *Consider a pseudo-Riemannian manifold of the form $(N \times \mathbb{R}, h)$, let $\widetilde{\nabla}$ denote the Levi-Civita connection of h (which need not be a product metric), let s denote the standard coordinate on $\mathbb{R}$, and suppose that $\partial_s \in \Gamma(T(N \times \mathbb{R}))$ is $\widetilde{\nabla}$-parallel. Suppose $\chi \in \Gamma(\bigotimes^r T^*(N \times \mathbb{R})|_{N\times\{0\}})$ is parallel with respect to the restriction of $\widetilde{\nabla}$. Then, the tensor $\widetilde{\chi} \in \Gamma(\bigotimes^r T^*(N \times \mathbb{R}))$ defined by translation of χ in the $\mathbb{R}$-direction is $\widetilde{\nabla}$-parallel.*

Proof of Proposition 2.1.9. If $\lambda \neq 0$, then in the coordinates (u, v) defined by $u = t(1+\lambda\rho)$, $v = t(1-\lambda\rho)$, the canonical ambient metric has the form

$$\widetilde{g}^{\mathrm{can}} = \frac{1}{2\lambda}(du^2 - dv^2) + u^2 g.$$

This is a product decomposition of $\widetilde{g}$, where one factor is a (negative-definite) line with coordinate v, so ∂_v is $\widetilde{\nabla}$-parallel, where $\widetilde{\nabla}$ is the Levi-Civita connection of $\widetilde{g}^{\mathrm{can}}$. Now, in coordinates $(u, s) = (u, v-u)$, $\mathcal{G} = \{s = 0\}$ and $\partial_s = \partial_v$, so the lemma yields a parallel extension of χ.

Now, if $\lambda = 0$, then in the coordinates (t, s) defined by $s = \rho t$, the canonical ambient metric is

$$\widetilde{g}^{\mathrm{can}} = 2\, dt\, ds + t^2 g.$$

Computing shows that ∂_s is $\widetilde{\nabla}$-parallel, and $\mathcal{G} = \{s = 0\}$, so the lemma again yields an extension.

□

Remark 2.1.11. Since any parallel extension to $\widetilde{M}$ is unique, the tensor $\widetilde{\chi}$ produced in the previous proposition agrees with the ambient extension $\widetilde{\chi}$ constructed in the proof of Theorem 2.1.2 to infinite order.

Proof of Lemma 2.1.10. Since ∂_s is parallel, it is Killing, so h and hence $\widetilde{\nabla}$ are invariant under the flow of ∂_s, which is just given by the translations $\mu_a(x, s) := (x, s + a)$. So, for any $Y \in T(N \times \mathbb{R})$, say, with base point (x, s),

$$\begin{aligned}(\widetilde{\nabla}_Y \widetilde{\chi})_{(x,s)} &= (\mu^*_{-s}\widetilde{\nabla})_Y (\mu^*_{-s}\widetilde{\chi})_{(x,s)} \\ &= \mu^*_{-s}(\widetilde{\nabla}_{d\mu_{-s}(Y)}\widetilde{\chi})_{\mu_{-s}(x,s)} \\ &= \mu^*_{-s}(\widetilde{\nabla}_{d\mu_{-s}(Y)}\widetilde{\chi})_{(x,0)}.\end{aligned}$$

Thus, to show that $\widetilde{\chi}$ is $\widetilde{\nabla}$-parallel, it suffices to show that $\widetilde{\nabla}_Y \widetilde{\chi}$ just for $Y \in T(N \times \mathbb{R})$ with base points $(x, 0)$. Now, since χ is parallel and $\widetilde{\chi}$ extends χ, for $Z \in T(N \times \{0\})$ we have

$$\widetilde{\nabla}_Z \widetilde{\chi} = \widetilde{\nabla}_Z \chi = 0.$$

Because $\partial_s|_{(y,0)}$ and $T_{(y,0)}(N \times \{0\})$ span $T_{(y,0)}(N \times \mathbb{R})$ for all $y \in N$, to prove that $\widetilde{\nabla}\widetilde{\chi} = 0$ it only remains show that $\widetilde{\nabla}_{\partial_s}\widetilde{\chi} = 0$. Now, by definition $\widetilde{\chi}$ is invariant under the flow of ∂_s, that is, $\mathcal{L}_{\partial_s}\widetilde{\chi} = 0$. Using the index notation formula for the Lie derivative, $\mathcal{L}$, in terms of a torsion-free connection gives

$$0 = (\mathcal{L}_{\partial_s}\widetilde{\chi})_{a_1\cdots a_r} = (\partial_s)^b \widetilde{\chi}_{a_1\cdots a_r,b} + \sum_{u=1}^{r} (\partial_s)^b{}_{,a_u} \widetilde{\chi}_{a_1\cdots a_{u-1}ba_{u+1}\cdots a_r}.$$

By hypothesis ∂_s is parallel, so each term in the summation is zero, leaving the desired equation $\widetilde{\nabla}_{\partial_s}\widetilde{\chi} = 0$. □

Remark 2.1.12. The explicit form of the ambient metric of a Einstein conformal structure c makes it possible to compute the ambient extensions of tractor tensors for such structures.

For example, the parallel tractor r-form whose representation with respect to the Einstein representative $g \in c$ is

$$\begin{pmatrix} \chi_-, & \begin{matrix}\chi_0 \\ \chi_\mp\end{matrix} & , \chi_+ \end{pmatrix} \tag{2.1}$$

has parallel extension

$$\widetilde{\chi}(t,x,\rho) = t^{r-1}(1+\lambda\rho)^{r-1} dt \wedge (\chi_- + \rho\chi_+) + t^r(1+\lambda\rho)^r \chi_0 \\ + t^{r-1}(1+\lambda\rho)^{r-2} dt \wedge d\rho \wedge \chi_{\mp} + t^r(1+\lambda\rho)^{r-1} d\rho \wedge \chi_+.$$

The following proposition shows that at least on obstruction-flat manifolds, the extensibility of a parallel tractor tensor to infinite order is obstructed only at the critical order $\frac{n}{2} - 1$.

Proposition 2.1.13. *Let (M, c) be an obstruction-flat conformal manifold with even dimension n, and let $\widetilde{g}$ be an infinite-order ambient metric for c. If a parallel tractor tensor $\chi \in \Gamma(\bigotimes^r \mathcal{T}^*)$ has an ambient extension $\widetilde{\chi}'$ satisfying $\widetilde{\nabla}\widetilde{\chi}' = O(\rho^{n/2})$, then it has an ambient extension $\widetilde{\chi}$ satisfying $\widetilde{\chi} = O(\rho^\infty)$.*

Proof. Let $\widetilde{\chi}$ be the ambient extension of χ defined in the proof of the Parallel Tractor Extension Theorem. Then, by construction, $\widetilde{\chi} - \widetilde{\chi}' = O(\rho)$ and $\widetilde{\nabla}_{\partial_\rho}(\widetilde{\chi} - \widetilde{\chi}') = O(\rho^{n/2})$. Expanding the left hand side of the second equation in Christoffel symbols, inductively differentiating with respect to ρ $n/2$ times and evaluating at $\rho = 0$ at each step yields $\widetilde{\chi} - \widetilde{\chi}' = O(\rho^{n/2+1})$, and thus $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2})$. The induction step in the proof of Theorem 2.1.2 works for $s \neq \frac{n}{2} - 1$, so applying the induction argument to $\widetilde{\nabla}\widetilde{\chi}$ beginning at order $\frac{n}{2}$ gives that $\widetilde{\nabla}\widetilde{\chi} = O(\rho^\infty)$. (The induction uses Lemma 2.1.4, which by the comment after its proof applies in this context, because c is obstruction-flat.) □

2.1.2 Holonomy reduction of ambient metrics

If a conformal structure c, say, with signature (p, q), admits a parallel tractor χ, then the conformal holonomy of c reduces to a subgroup of the stabilizer of χ at a point. So, when there exists a bona fide parallel ambient extension $\widetilde{\chi}$ of χ with respect to an ambient metric $\widetilde{g}$ for c, which occurs at least when n is odd and c is real-analytic (by Theorem 2.1.3) or c is Einstein (by Proposition 2.1.9), then the holonomy of the ambient metric is reduced to a subgroup of $\mathrm{O}(p+1, q+1)$. In the case that c is Einstein, the ambient holonomy reduces as follows. Compare this result with Proposition 3.1.14, which gives the specialization of this proposition to conformal structures induced by generic 2-plane fields on 5-manifolds.

Proposition 2.1.14. *Let (M, c) be an Einstein conformal structure of signature (p, q), say with Einstein representative $g \in c$ and $\operatorname{Ric} g = 2(n-1)\lambda g$, let $\xi \in \Gamma(\mathcal{T})$ be the corresponding parallel tractor, let $\widetilde{g}^{\mathrm{can}}$ be the canonical ambient metric defined in (1.21), and let Ξ be the unique $\widetilde{g}$-parallel ambient extension of ξ. Then,*

- *If $\lambda < 0$, then Ξ is spacelike, and $\operatorname{Hol}(\widetilde{g}^{\mathrm{can}}) \leq \mathrm{O}(p, q+1)$.*

- *If $\lambda = 0$ (i.e., g is Ricci-flat), then Ξ is lightlike, and $\operatorname{Hol}(\widetilde{g}^{\mathrm{can}})$ is contained in the stabilizer in $\emptyset(p+1, q+1)$ of a null vector in $\mathbb{R}^{p+1,q+1}$.*

- *If $\lambda > 0$, then Ξ is timelike, and $\operatorname{Hol}(\widetilde{g}^{\mathrm{can}}) \leq \mathrm{O}(p+1, q)$.*

If $\lambda \neq 0$, then any point in $\widetilde{M}$ admits a neighborhood on which $\widetilde{g}$ decomposes as a pseudo-Riemannian product metric $(V, k) \times (U, h)$ such that $TV = \operatorname{span}\{\Xi\}|_V$ and $TU = \{\Xi\}^{\perp}|_U$.

Proof. Using the formula after Proposition 1.3.6 gives $g^{\mathcal{T}}(\xi, \xi) = -2\lambda$ (or, equivalently, $\widetilde{g}^{\mathrm{can}}(\Xi, \Xi) = -2\lambda$). The respective stabilizers in $\mathrm{O}(p+1, q+1)$ of a spacelike and a timelike vector in $\mathbb{R}^{p+1,q+1}$ are, respectively, $\mathrm{O}(p, q+1)$ and $\mathrm{O}(p+1, q)$.

Since Ξ is parallel, so are $\operatorname{span}\{\Xi\}$ and $\{\Xi\}^{\perp}$, and if $\lambda \neq 0$, then $\operatorname{span}\{\Xi\} \cap \{\Xi\}^{\perp} = \{0\}$. Pick an arbitrary point $x \in \widetilde{M}$; by construction, $\operatorname{span}\{\Xi_x\}$ and $\{\Xi_x\}^{\perp}$ are $\operatorname{Hol}_x(\widetilde{M}, \widetilde{g}^{\mathrm{can}})$-submodules; decompose the latter into a direct sum $\bigoplus_j E_j$ of indecomposable submodules. Then, $\operatorname{span}\{\Xi_x\} \oplus \bigoplus_j E_j$ is a decomposition of $T_x\widetilde{M}$ into indecomposable $\operatorname{Hol}_x(\widetilde{M}, \widetilde{g}^{\mathrm{can}})$-submodules, so the de Rham Theorem (Theorem 1.6.2) yields a decomposition of some neighborhood of x into a pseudo-Riemannian product manifold $(V, k) \times (\prod_j U_j, \bigoplus_j h_j)$. Denote $U := \prod U_j$; by construction, $TV = \operatorname{span}\{\Xi\}|_V$ and $TU = \{\Xi\}^{\perp}|_U$. (This product decomposition result can be proven more efficiently using another form of de Rham's Theorem, for example, [Wu64, Section 2].) □

These results are essentially contained in those cited for Proposition 2.1.9, and the observation about the relationship between the sign of the Einstein constant, λ, and the sign of $g^{\mathcal{T}}(\xi, \xi)$ is implicit in [Gov, Section 2].

2.2 *The critical order for n even*

In this section, (M, c) is a conformal structure of even dimension $n > 2$. Recall that in this case, a conformal structure only determines an ambient metric up to diffeomorphism and up to an ambiguity at order $n/2$, and that a choice of this ambiguity determines the ambient metric to infinite order. For an arbitrary parallel tractor tensor χ, Theorem 2.1.2 only guarantees the existence of an ambient extension parallel to order $\frac{n}{2} - 1$, and so the existence of an ambient extension parallel to infinite order a priori may depend on the choice of the ambiguity. In this section, we develop some criteria for uniqueness of an ambient metric for which χ admits an ambient extension parallel to infinite order.

2.2.1 Determining tractor tensors

In this section we restrict attention to obstruction-flat conformal structures, by loosening a hypothesis in the formulation of the ambient metric, one can extend this investigation to non–obstruction-flat structures; see Section 3.3 for more discussion of this point. Recall that in this case, Proposition 2.1.13 says that if a parallel tractor tensor χ admits an extension parallel to order $\frac{n}{2}$, then it admits an extension parallel to infinite order.

We formulate a condition on parallel tractor tensors χ that guarantees uniqueness of an infinite-order ambient metric (see Subsection 1.2.5) for which χ admits a parallel extension, but not necessarily existence. Recall that 1-forms on M can be embedded naturally in the set of tractor 2-forms just by insertion into the injecting part, or by raising an index with respect to $g^{\mathcal{T}}$, in the space of adjoint tractors (that is, $g^{\mathcal{T}}$-skew tractor endomorphisms), $\Gamma(\mathcal{A}M)$. Explicitly, let $\pi : \mathcal{G} \to M$ denote the natural projection. Then, for any $\eta \in T^*_x M$, pulling back to the metric bundle yields a section $\pi^*\eta \in \Gamma(T^*\mathcal{G}|_{\pi^{-1}(x)})$ that annihilates $\mathbb{T}^A$, and since $\mathbb{T}_A$ spans the annihilator of $T\mathcal{G} \subset T\widetilde{M}$, we may regard $\pi^*\eta$ as an element of $\mathcal{T}^*_x(-1) \bmod \mathbb{T}_A$. So, define a bundle map $\iota : T^*M \to \mathcal{A}M$ by

$$\iota(\eta)^A{}_B := 2(g^{\mathcal{T}})^{AC}\mathbb{T}_{[C}(\pi^*\eta)_{B]},$$

where $\pi^*\eta$ is any representative of $\mathcal{T}^*_x(-1) \bmod \mathbb{T}_A$; because this quantity is skewed with $\mathbb{T}_A$, $\iota(\eta)^A{}_C$ is independent of this ambiguity. With respect to the splitting induced by any choice of representative $g \in c$,

$$\iota(\eta)^A{}_B = \begin{pmatrix} 0 & \eta_b & 0 \\ 0 & 0 & -\eta^a \\ 0 & 0 & 0 \end{pmatrix}.$$

The map ι is an adjoint analogue of the natural injection $\mathcal{D}[-1] \hookrightarrow \mathcal{T}$ defined in Subsection 1.3.1. Now, for any tractor tensor $\chi \in \Gamma(\bigotimes^r \mathcal{T}^*)$ define another bundle map, $F_\chi : T^*M \to \bigotimes^r \mathcal{T}^*$, by

$$F_\chi(\eta) := \iota(\eta).\chi,$$

where . denotes the natural action of $\mathcal{A}M$ on $\bigotimes^r \mathcal{T}^*$; explicitly,

$$[F_\chi(\eta)]_{\mathbf{A}} = -\sum_{u=1}^{r} \iota(\eta)^Q{}_{A_u} \chi_{A_1 \cdots A_{u-1} Q \cdots A_{u+1} \cdots A_r}.$$

Definition 2.2.1. A tractor tensor χ is **determining** if the induced map $F_\chi : \Gamma(T^*M) \to \Gamma(\bigotimes^r \mathcal{T}^*)$ on sections is injective; a tractor is **nondetermining** if it is not determining.

The primary purpose for this definition is the following consequence:

Proposition 2.2.2. *Suppose $n > 2$ is even and that c is obstruction-flat. Suppose χ is parallel and determining. Then, there is at most one infinite-order ambient metric for which χ admits an ambient extension satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2})$. (If there is such an ambient metric, Proposition 2.1.13 shows that χ moreover admits an extension satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^\infty)$ for that metric.)*

Proof. Put $\widetilde{g}$ into normal form with respect to an arbitrary representative $g \in c$. If χ admits an ambient extension satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2})$, then commuting covariant derivatives and distributing the covariant derivative indices as in the proof of the Parallel Tractor Extension Theorem gives

$$\begin{aligned} 0 &= \widetilde{\chi}_{\mathbf{A},b\underbrace{\infty\cdots\infty}_{n/2-1}} - \widetilde{\chi}_{\mathbf{A},\infty b\underbrace{\infty\cdots\infty}_{n/2-2}} \\ &= -\left(\sum_{u=1}^{r} \widetilde{R}_{b\infty}{}^Q{}_{A_u} \widetilde{\chi}_{A_1\cdots A_{u-1}QA_{u+1}\cdots A_r}\right)_{,\underbrace{\infty\cdots\infty}_{n/2-2}} \\ &= -\sum_{u=1}^{r} \widetilde{R}_{b\infty}{}^Q{}_{A_u,\underbrace{\infty\cdots\infty}_{n/2-2}} \widetilde{\chi}_{A_1\cdots A_{u-1}QA_{u+1}\cdots A_r}. \end{aligned}$$

The only components of curvature that appear in the sum that depend on the ambiguity in the ambient metric are those of the form $\widetilde{R}_{b\infty}{}^{0}{}_{a}$ and $\widetilde{R}_{b\infty}{}^{q}{}_{\infty}$; the others are determined by g.

Henceforth, we restrict to $M \subset \widetilde{M}$. For an arbitrary vector field $V \in \Gamma(TM)$, define $\eta_a = V^b \widetilde{R}_{\infty ab\infty,\underbrace{\infty\cdots\infty}_{n/2-2}}$. Unwinding the definitions, $\iota(\eta)^Q{}_A = -V^b \widetilde{R}_{\infty b}{}^{Q}{}_{A,\underbrace{\infty\cdots\infty}_{n/2-2}}$ for ${}^Q{}_A = {}^0{}_a$ and ${}^Q{}_A = {}^q{}_\infty$, and $\iota(\eta)^Q{}_A = 0$ otherwise, so $\iota(\eta)^Q{}_A$ comprises exactly the part of $-V^b \widetilde{R}_{\infty b}{}^{Q}{}_{A,\underbrace{\infty\cdots\infty}_{n/2-2}}$ that depends on the ambiguity in the ambient metric, so $D^Q{}_A :=$ $-V^b \widetilde{R}_{\infty b}{}^{Q}{}_{A,\underbrace{\infty\cdots\infty}_{n/2-2}} - \iota(\eta)^Q{}_A$ is independent of the ambiguity in the ambient metric. Contracting V into the previous display equation gives $-V^b \sum_{u=1}^{r} \widetilde{R}_{b\infty}{}^{Q}{}_{A_u,\underbrace{\infty\cdots\infty}_{n/2-2}} \widetilde{\chi}_{A_1\cdots A_{u-1}QA_{u+1}\cdots A_r} =$ 0, which can be rewritten as $F_\chi(\eta) = -D.\chi$. Since χ is determining, at most one choice of η satisfies this equation. Since the choice of V is arbitrary, there is at most possibility of $\widetilde{R}_{\infty ab\infty,\underbrace{\infty\cdots\infty}_{n/2-2}}$; this tensor parameterizes the ambiguity in the ambient metric, yielding the desired uniqueness. □

Tractor tensors sometimes fail to be determining.

Example 2.2.3. By definition, for $\chi \in \Gamma(\bigotimes^0 \mathcal{T}^*) \cong \mathcal{E}$ we have $F_\chi(\cdot) = \iota(\cdot).\chi = 0$, so functions are never determining.

Example 2.2.4. Example 2.1.7 observes that $g^{\mathcal{T}}$ admits parallel extensions for more than one choice—in fact, every choice—of ambient metric, so it is not determining. Indeed, $\beta.g^{\mathcal{T}} = 0$ for every adjoint tractor β.

Proposition 2.2.2 can be plied to show in several important cases the uniqueness of an ambient metric $\widetilde{g}$ (in the sense described in the proposition) for which a parallel tractor tensor admits a parallel ambient extension. In some cases, we will need explicit formulas for the components of F_χ with with respect to the splitting induced by an arbitrary choice of representative $g \in c$. For example, if χ is a tractor 1-form, then

$$[F_\chi(\eta)]_A = (0, -\eta_a \chi_0, \eta^q \chi_q), \tag{2.2}$$

and if χ is a tractor r-form, $r > 1$, then

$$(F_\chi(\eta))_{\mathbf{A}} = \left(0, \quad \begin{matrix} -r\eta_{[a_1}\chi_{|0|a_2\cdots a_r]} \\ \eta^q\chi_{0qa_3\cdots i_a} \end{matrix} \quad , \eta^q\chi_{qa_2\cdots a_r} + (r-1)\eta_{[a_2}\chi_{|0\infty|a_3\cdots a_r]}\right) \in \Gamma\left(\Lambda^r\mathcal{T}^*\right). \tag{2.3}$$

If $\mathbb{W}$ is an irreducible $\mathrm{O}(p+1, q+1)$-representation and W is the tractor tensor bundle it induces, by Theorem 1.3.14 a parallel section $\chi \in \Gamma(W)$ can be recovered from its projecting part $\alpha := \Pi_0(\chi)$ using the BGG splitting operator L_0 for W. (Recall that Theorem 1.3.14 strictly applies to a larger parabolic subgroup of $\mathrm{O}(p+1, q+1)$ with the same subalgebra $\dot{\mathfrak{p}}$ of $\dot{P}$, but that result appears to be true for $\dot{P}$, too; we henceforth assume that it is.) So, one can write F_χ in terms of α, that is, as $F_{L_0(\alpha)}$, and investigate the determining criterion in terms of that typically simpler object.

Example 2.2.5. If $\mathbb{W} = \square$ (the standard representation), so that χ is a parallel tractor 1-form and its projecting part is $\alpha := \chi_0 = \Pi_0(\chi)$, then composing (2.2) with the BGG splitting operator (1.39) gives

$$[F_\chi(\eta)]_A = (0, -\eta_a\alpha, \eta^q\alpha_q), \tag{2.4}$$

Example 2.2.6. If $\mathbb{W} = \left.\begin{matrix}\square \\ \vdots \\ \square\end{matrix}\right\} r$, so that χ is a tractor r-form and its projecting part is $\alpha_{a_2\cdots a_r} := \chi_{0a_2\cdots a_r}$, then composing (2.3) with the BGG splitting operator (1.41) gives

$$(F_\chi(\eta))_{\mathbf{A}} = \left(0, \quad \begin{matrix} -r\eta_{[a_1}\alpha_{a_2\cdots a_r]} \\ \eta^q\alpha_{qa_3\cdots a_r} \end{matrix} \quad , \eta^q\alpha_{[a_2\cdots a_r,q]} - \tfrac{r-1}{n-r+2}\eta_{[a_2}\alpha_{|q|a_3\cdots a_r],}{}^{q}\right) \in \Gamma\left(\Lambda^r\mathcal{T}^*\right). \tag{2.5}$$

Recall from Proposition 1.3.8 that a nonzero parallel cotractor χ defines an almost Einstein scale, and from Proposition 1.3.9 that its projecting part $\chi_0 = \Pi_0(\chi)$ is nonzero on a dense subset of M.

Proposition 2.2.7. *Any nonzero parallel tractor $\chi \in \Gamma(\mathcal{T}^*)$ is determining.*

Proof. By (2.4), $[F_\chi(\eta)]_a = -\eta_a\chi_0$. Since χ is parallel and nonzero Proposition 1.3.8 gives that χ_0 is an almost Einstein scale, and so by Proposition 1.3.9 it is nonzero on a dense subset of M. Thus, if $[F_\chi(\eta)] = 0$ then $\eta = 0$, that is, χ is determining. $\square$

Together with Example 2.1.8, this proposition immediately yields the following:

Corollary 2.2.8. *Suppose $n > 2$ is even and that c contains an Einstein representative g. Up to infinite order and diffeomorphism fixing $\mathcal{G}$ pointwise, the canonical ambient metric $\widetilde{g}^{\mathrm{can}}$ is the unique infinite-order ambient metric for c for which the Einstein scale χ admits an extension $\widetilde{\chi}$ satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2})$. In particular, if c is real-analytic, then the canonical ambient metric $\widetilde{g}^{\mathrm{can}}$ is the unique real-analytic ambient metric (up to diffeomorphism fixing $\mathcal{G}$ pointwise and up to extension) for which χ admits a parallel ambient extension.*

Higher-rank parallel tractor tensors exhibit more varied behaviors. We first consider parallel tractor r-forms χ and then first the special case $r = 2$. As above, we denote the projecting part of a parallel tractor r-form by $\alpha_{a_2 \cdots a_r} := \chi_{0 a_2 \cdots a_r}$.

Proposition 2.2.9. *A parallel tractor $\chi \in \Gamma(\Lambda^2 \mathcal{T}^*)$ is determining iff its projecting part is non-null on a dense open subset of M.*

Proof. As before, denote the projecting part of χ by $\alpha := \Pi_0(\chi)$. If χ vanishes anywhere, then it is the zero tractor tensor; so, trivially, it is nondetermining and $\alpha_q \alpha^q = 0$. Thus, assume henceforth that χ is nonvanishing; in particular, recall from Proposition 1.3.18 that α is nonzero on a dense subset of M.

Suppose that $F_\chi(\eta) = 0$. Equation (2.5) gives that $0 = [F_\chi(\eta)]_{ab} = -2\eta_{[a}\alpha_{b]}$. Since α_b is nonzero on a dense open subset $U \subseteq M$, $\eta_a|_U = f\alpha_a|_U$ for some function $f \in C^\infty(U)$. Consulting (2.5) again gives that $0 = [F_\chi(\eta)]_{0\infty}|_U = (\eta^q \alpha_q)|_U = f(\alpha_q\alpha^q)|_U$.

So, if $\alpha_q\alpha^q$ is nonvanishing on a dense subset $V \subset M$, then f is identically zero on the dense subset $U \cap V \subseteq M$. Thus, $\eta_a|_V = f|_V \alpha_a|_V = 0$, and by continuity, $\eta_a = 0$, so χ is determining.

Conversely, suppose $\alpha_q\alpha^q$ vanishes on some nonempty, open set W, and let ψ be a bump function supported in W. Since α is nonzero on a dense subset of M, $\psi\alpha \neq 0$; we show that $F_\chi(\psi\alpha) = 0$, which gives that χ is not determining, by showing separately that its components $F_\chi(\psi\alpha)_{AB}$ vanish. Consulting (2.5) gives that $[F_\chi(\psi\alpha)]_{0a} = 0$, and substituting gives $[F_\chi(\psi\alpha)]_{ab} = -2\psi\alpha_{[a}\alpha_{b]} = 0$. Also, $[F_\chi(\psi\alpha)]_{0\infty} = \psi\alpha_q\alpha^q$; this vanishes on W by definition, and it vanishes on the complement of W because ψ does. It remains to check that $[F_\chi(\psi\alpha)]_{\infty a} = 0$. Substituting in the formula from (2.5) gives

$$[F_\chi(\psi\alpha)]_{\infty a} = (\psi\alpha^q)\alpha_{[a,q]} - \tfrac{1}{n}(\psi\alpha_a)\alpha_{q,}{}^q = \psi\left(\tfrac{1}{2}\alpha^q\alpha_{a,q} - \tfrac{1}{2}\alpha^q\alpha_{q,a} - \tfrac{1}{n}\alpha_a\alpha_{q,}{}^q\right).$$

Since α is null, differentiating gives $0 = (\alpha_q\alpha^q)_{,a} = 2\alpha^q\alpha_{q,a}$, so the second term in the parentheses in the rightmost expression vanishes. Contracting α into the conformal Killing form equation $\Theta_0(\alpha) = 0$ (see (1.42)) gives $\alpha^q\alpha_{a,q} - \frac{2}{n}\alpha_a\alpha_{q,}{}^q = 0$, so the remaining terms in those parentheses vanish, and so $F_\chi(\psi\alpha)$ vanishes as desired. □

We can give a similar result for all (not necessarily parallel) tractor r-forms, for all $r \geq 2$. As before, let α denote the projecting part of χ.

Proposition 2.2.10. *Suppose $\chi \in \Gamma(\Lambda^r\mathcal{T}^*)$, $r \geq 2$ is nondetermining. Then, at least one of the following holds:*

1. *The projecting part α vanishes on some nonempty open set.*

2. *There is a nonempty open set U such that $\alpha|_U = \beta \wedge \gamma$, where $\beta \in \Gamma(T^*U)$ is null and $\gamma \in \Gamma(\Lambda^{r-2}T^*U)$.*

In particular, if the conformal structure c is definite, then (1) holds. If instead χ is parallel and nonzero, (2) holds.

Proof. By definition, there is some nonzero $\eta \in \Gamma(T^*M)$ such that $F_\chi(\eta) = 0$. Then, the formulas for the $a_1 \cdots a_r$ and $0\infty a_3 \cdots a_r$ components of $F_\chi(\eta)$ in Equation (2.5) give respectively that $\eta \wedge \alpha$ and $\eta^\sharp \lrcorner\, \alpha$ both vanish identically. Then,

$$0 = \eta \wedge (\eta^\sharp \lrcorner\, \alpha) + \eta^\sharp \lrcorner\, (\eta \wedge \alpha) = \langle \eta, \eta \rangle\, \alpha.$$

So, if (i) does not hold, then η is null on a dense subset of M and hence, by continuity, everywhere. Let U be the (nonempty) set on which η is nonzero. Then, $\eta \wedge \alpha = 0$ implies that $\alpha|_U = \eta|_U \wedge \gamma$ for some $\gamma \in \Gamma(\Lambda^{r-2}T^*U)$. Taking $\beta = \eta|_U$ satisfies the hypothesis.

If χ is also parallel and nonzero, then by Proposition 1.3.18, α is nonvanishing on a dense subset of M, eliminating the possibility of (1). □

This proposition, together with Proposition 2.2.7, gives the following result in the special case that c is definite.

Corollary 2.2.11. *Suppose c is definite. Then, any nonzero parallel tractor r-form, $\chi \in \Gamma(\Lambda^r \mathcal{T}^*)$, $r \geq 1$, is determining.*

We produce analogs of the above results for sections of tractor tensor bundles induced by irreducible $\mathrm{O}(p+1, q+1)$-representations. To do so, we use the fact that, because F_χ is defined by an endomorphism action, it obeys a corresponding Leibniz rule.

Proposition 2.2.12. *The operator F_χ satisfies*

$$F_{\chi^{(1)} \otimes \cdots \otimes \chi^{(s)}}(\eta) = \sum_{v=1}^{s} \chi^{(1)} \otimes \cdots \otimes \chi^{(v-1)} \otimes [F_{\chi^{(v)}}(\eta)] \otimes \chi^{(v+1)} \otimes \cdots \otimes \chi^{(s)}.$$

Proof. Applying the definition,

$$\begin{aligned} F_{\chi^{(1)} \otimes \cdots \otimes \chi^{(s)}}(\eta) &= \iota(\eta).(\chi^{(1)} \otimes \cdots \otimes \chi^{(s)}) \\ &= \sum_{v=1}^{s} \chi^{(1)} \otimes \cdots \otimes \chi^{(v-1)} \otimes [\iota(\eta).\chi^{(v)}] \otimes \chi^{(v+1)} \otimes \cdots \otimes \chi^{(s)} \\ &= \sum_{v=1}^{s} \chi^{(1)} \otimes \cdots \otimes \chi^{(v-1)} \otimes [F_{\chi^{(v)}}(\eta)] \otimes \chi^{(v+1)} \otimes \cdots \otimes \chi^{(s)}. \end{aligned}$$

□

Let $\lambda = (r_1, \ldots, r_s)$ be a partition of some nonnegative integer r with $r_1 + r_2 \leq n$; recall from Subsection 1.2.1 that we denote the corresponding $\mathrm{O}(p+1, q+1)$-representation by $\mathbb{S}_{[\lambda]}(\mathbb{R}^{p+1,q+1})^*$. For $j = 1, \ldots, s$, let $\mathbf{A}^{(j)}$ denote the multi-index $A_1^{(j)} \cdots A_{r_j}^{(j)}$. Then, we may write any section of the corresponding tractor tensor bundle,

$$W := E \times_{\dot{P}} \mathbb{W} \subseteq \bigotimes_{j=1}^{s} \Lambda^{r_j} \mathcal{T}^* \subseteq \bigotimes^{r} \mathcal{T}^*$$

(see Subsection 1.3.4) as $\chi_{\mathbf{A}^{(1)} \cdots \mathbf{A}^{(s)}}$, in particular so that for each j, χ is antisymmetric in the indices constituting $\mathbf{A}^{(j)}$, which correspond to the jth column of the Young diagram defined by λ.

Examples 1.3.11 and 1.3.12 together with further experimentation not recorded here suggests that for a general nontrivial irreducible $\mathrm{O}(p+1,q+1)$-representation $\mathbb{W}$, the induced canonical projection $W \to W_{\dot{\mathfrak{n}}}$ can be identified with the map

$$\chi_{\mathbf{A}^{(1)}\cdots\mathbf{A}^{(s)}} \mapsto \chi_{0a_2^{(1)}\cdots a_{r_1}^{(1)}\cdots 0a_2^{(s)}\cdots a_{r_s}^{(s)}}$$

where we regard the right-hand side as an element of $(\bigotimes_{j=1}^{s}\Lambda^{r_j-1}T^*M)[r]$. Equivalently, this is the map given by contracting $\chi_{\mathbf{A}^{(1)}\cdots\mathbf{A}^{(s)}}$ with $\mathbb{T}^{A_1^{(1)}}\cdots\mathbb{T}^{A_1^{(s)}}$ and pulling back by the inclusion $\mathcal{G} \hookrightarrow \widetilde{M}$. For any Young diagram Y (necessarily marked with a subscript 0), call the statement that this map is indeed the canonical projection for the corresponding irreducible $\mathrm{O}(p+1,q+1)$-representation $\mathbb{W}$ the *tractor projection hypothesis* for Y.

Proposition 2.2.13. *Suppose $\lambda = (r_1,\ldots,r_s)$ is a partition for which $r_s = 1$. If χ is a nonzero parallel section of the bundle W induced by the irreducible* $\mathrm{O}(p+1,q+1)$*-representation $\mathbb{S}_{[\lambda]}(\mathbb{R}^{p+1,q+1})^*$, then χ is determining.*

Proof. Suppose that the Young diagram has column heights $r_1 \geq \cdots \geq r_s = 1$. First suppose that χ can be written as

$$\chi = \chi^{(1)} \otimes \cdots \otimes \chi^{(s)}, \tag{2.6}$$

where $\chi^{(u)} \in \Gamma(\Lambda^{r_u}\mathcal{T}^*)$ for $1 \leq u \leq s$. (Any section $\chi \in \Gamma(W)$ can be written as a linear combination of such tensors.)

For $1 \leq u \leq s-1$ denote by $\mathbf{A}_-^{(u)}$ the multi-index $0a_2^{(u)}\cdots a_{r_u}^{(u)}$. We compute the component $F_\chi(\eta)_{\mathbf{A}_-^{(1)}\cdots\mathbf{A}_-^{(s-1)}A}$: By Proposition 2.2.12,

$$\begin{aligned} F_\chi(\eta)_{\mathbf{A}_-^{(1)}\cdots\mathbf{A}_-^{(s-1)}A} &= \left(\sum_{u=1}^{s-1}\chi^{(1)}_{\mathbf{A}_-^{(1)}}\cdots\chi^{(u-1)}_{\mathbf{A}_-^{(u-1)}}[F_{\chi^{(u)}}(\eta)]_{\mathbf{A}_-^{(u)}}\chi^{(u+1)}_{\mathbf{A}_-^{(u+1)}}\cdots\chi^{(s-1)}_{\mathbf{A}_-^{(s-1)}}\chi^{(s)}_A\right) \\ &\qquad + \chi^{(1)}_{\mathbf{A}_-^{(1)}}\cdots\chi^{(s-1)}_{\mathbf{A}_-^{(s-1)}}[F_{\chi^{(s)}}(\eta)]_A. \end{aligned}$$

Now, each factor $[F_{\chi^{(u)}}(\eta)]_{\mathbf{A}_-^{(u)}}$ vanishes by equation (2.2) (if $r_u = 1$) or (2.3) (if $r_u > 1$), leaving only the last term:

$$F_\chi(\eta)_{\mathbf{A}_-^{(1)}\cdots\mathbf{A}_-^{(s-1)}A} = \chi^{(1)}_{\mathbf{A}_-}\cdots\chi^{(s-1)}_{\mathbf{A}_-}[F_{\chi^{(s)}}(\eta)]_A.$$

By (2.2), $[F_{\chi^{(s)}}(\eta)]_a = \eta_a \chi_0^{(s)}$, so

$$F_\chi(\eta)_{\mathbf{A}_-^{(1)}\cdots\mathbf{A}_-^{(s-1)}a} = \eta_a \chi_{\mathbf{A}_-^{(1)}\cdots\mathbf{A}_-^{(s-1)}0}.$$

By linearity, the same holds for arbitrary tractors χ, not just those admitting the decomposition (2.6). Now, $\chi_{\mathbf{A}_-^{(1)}\cdots\mathbf{A}_-^{(s-1)}0}$ is the projecting part of χ; since it is parallel, Proposition 1.3.18 gives that it is nonzero on some dense subset $U \subseteq M$. So, if $F_\chi(\eta) = 0$, then η is zero on U and by continuity on all of M, that is, χ is determining. □

Proposition 2.2.14. *Suppose that the tractor projection hypothesis holds for a partition $\lambda = (r_1, \ldots, r_s)$ of $r > 0$ with $r_1 + r_2 \leq n$ and that c is definite. If χ is a nonzero parallel section of the bundle W induced by the irreducible $\mathrm{O}(p+1, q+1)$-representation $\mathbb{S}_{[\lambda]}(\mathbb{R}^{p+1,q+1})^*$, then χ is determining.*

Proof. We use the notation of the proof of the previous proposition. Write χ in index notation as $\chi_{\mathbf{A}^{(1)}\cdots\mathbf{A}^{(s-1)}\mathbf{A}}$. If we pick local coordinates, fixing the multi-index $\mathbf{A}_-^{(1)} \cdots \mathbf{A}_-^{(s-1)}$ yields a tractor r_s-form $\mu \in \Gamma(\Lambda^{r_s}\mathcal{T}^*)$. Arguing as in the proof of Proposition 2.2.13 gives that $F_\mu(\eta)_{\mathbf{A}}$ is then simply $[F_\chi(\eta)]_{\mathbf{A}_-^{(1)}\cdots\mathbf{A}_-^{(s-1)}\mathbf{A}}$ with the multi-index $\mathbf{A}_-^{(1)} \cdots \mathbf{A}_-^{(s-1)}$ fixed.

Suppose χ is nondetermining, say that the nonzero 1-form $\eta \in \Gamma(T^*M)$ lies in $\ker F_\chi$, and let U be some nonempty open set on which η is nonvanishing. Since χ is nonzero and parallel, by Proposition 1.3.18 the projecting part $\chi_{\mathbf{A}_-^{(1)}\cdots\mathbf{A}_-^{(s-1)}\mathbf{A}_-}$ of χ is nonvanishing on some dense open subset on M, so we may replace U with some nonempty subset on which both η and $\chi_{\mathbf{A}_-^{(1)}\cdots\mathbf{A}_-^{(s-1)}\mathbf{A}_-}$ are nonvanishing. Replace U again with a coordinate chart on some subset of U; then, there is some multi-index $\mathbf{A}_-^{(1)} \cdots \mathbf{A}_-^{(s-1)}\mathbf{A}_-$ for which the component $\chi_{\mathbf{A}_-^{(1)}\cdots\mathbf{A}_-^{(s-1)}\mathbf{A}_-}$ is nonzero on some open subset of U, and we yet again replace U with this subset. Fix just the multi-index $\mathbf{A}_-^{(1)} \cdots \mathbf{A}_-^{(s-1)}$, which yields a tractor form μ as above. By construction, $F_\mu(\eta|_U) = 0$ but $\eta|_U \neq 0$ and hence μ is not determining. On the other hand, the projecting part $\mu_{\mathbf{A}_-}$ of μ is just the projecting part of χ with the multi-index $\mathbf{A}_-^{(1)} \cdots \mathbf{A}_-^{(s-1)}$ fixed, and thus it is nonvanishing on U. This contradicts Proposition 2.2.10, so χ must be determining. □

Remark 2.2.15. As in Proposition 2.2.10, in the case $r \geq 2$ we could have instead produced a stronger result, giving necessary conditions for η to be in $\ker F_\chi$, from which the stated result for c definite would follow; for parallel sections χ of bundles W corresponding to general Young diagrams, though, the analogous conditions on η are substantially more involved than for r-forms.

To illustrate how to analyze tractor tensor bundles induced by general (not necessarily irreducible) $\mathrm{O}(p+1, q+1)$-representations, we determine a necessary and sufficient condition for a general parallel 2-tensor to be determining. Recall that the decomposition of $\bigotimes^2 \mathcal{T}^*$ into irreducible $\mathrm{O}(p+1, q+1)$-subrepresentations is

$$\bigotimes^2 \mathcal{T}^* \cong \bullet \oplus \square\!\square_0 \oplus \begin{smallmatrix}\square\\ \square\end{smallmatrix}.$$

The projections of $\chi \in \Gamma(\bigotimes^2 \mathcal{T}^*)$ onto the corresponding subbundles are

$$\begin{aligned}
[\Pi_{\mathbb{R}}(\chi)]_{AB} &= (n+2)^{-1} \chi_Q{}^Q g^{\mathcal{T}}_{AB} \\
[\Pi_{S^2_0 \mathcal{T}^*}(\chi)]_{AB} &= \chi_{(AB)} - (n+2)^{-1} \chi_Q{}^Q g^{\mathcal{T}}_{AB} \\
[\Pi_{\Lambda^2 \mathcal{T}^*}(\chi)]_{AB} &= \chi_{[AB]}.
\end{aligned}$$

One can show that $\square\!\square_0$ satisfies the tractor projection hypothesis, so the projecting parts of the second two tractors are

$$\begin{aligned}
[\Pi_{S^2_0 \mathcal{T}^*}(\chi)]_{00} &= \chi_{00} \\
[\Pi_{\Lambda^2 \mathcal{T}^*}(\chi)]_{0b} &= \chi_{[0b]}.
\end{aligned}$$

Proposition 2.2.16. *Suppose χ is a parallel tractor 2-tensor. Then, χ is determining iff χ_{00} is nonzero or $\chi_{[0b]}$ is non-null on a dense subset.*

Proof. It suffices to show that $\Pi_{S^2_0 \mathcal{T}^*}(\chi)$ or $\Pi_{\Lambda^2 \mathcal{T}^*}(\chi)$ is determining.

Since $\square\!\square_0$ satisfies the tractor projection hypothesis, Proposition 2.2.13 gives that if $\chi_{00} = \Pi_{S^2_0 \mathcal{T}^*}(\chi)$ is not identically zero then χ is determining. Alternately, if $[\Pi_{\Lambda^2 \mathcal{T}^*}(\chi)]_{0b} = \chi_{[0b]}$ is non-null on a dense subset, then applying Proposition 2.2.9 shows that $\Pi_{\Lambda^2 \mathcal{T}^*}(\chi)$ is determining.

Suppose χ_{00} is identically zero and $\chi_{[0b]}$ is null on some nonempty open subset. Then, $\Pi_{S^2_0 \mathcal{T}^*} \chi = 0$ and so

$$F_\chi = F_{\Pi_{\mathrm{triv}} \chi} + F_{\Pi_{S^2_0 \mathcal{T}^*} \chi} + F_{\Pi_{\Lambda^2 \mathcal{T}^*} \chi} = F_{\Pi_{\Lambda^2 \mathcal{T}^*} \chi}$$

Since $[\Pi_{\Lambda^2\mathcal{T}^*}(\chi)]_{0b} = 2\chi_{[0b]}$ is null, by Proposition 2.2.9, $F_\chi = F_{\Pi_{\Lambda^2\mathcal{T}^*}\chi}$ is not injective, and so χ is not determining. $\square$

Chapter 3
APPLICATIONS

3.1 The ambient holonomy of Nurowski conformal structures

The group G_2 was among the last on Berger's List (see Theorem 1.6.4) to be realized as the holonomy group of a pseudo-Riemannian metric. Bryant indicated the construction [Bry87] of the first known example of a metric with G_2 holonomy, and Leistner and Nurowski constructed [LN10] the 8-parameter class of such metrics described in Example 1.6.13. In this section, we demonstrate that applying the Fefferman-Graham ambient construction to Nurowski's conformal structures produces a rich, new class of examples.

3.1.1 G_2 *holonomy*

Hammerl and Sagerschnig proved that a conformal structure induced by a generic 2-plane field on an orientable 5-manifold admits a certain kind of parallel tractor object. Call a (tractor or ambient) 3-form **split type** if its restriction to each point is split type (see Subsection 1.4.1). (If the 3-form is parallel with respect to the corresponding connection, to determine whether it is split type it suffices to check one point.)

Theorem 3.1.1. *[HS09] A conformal structure c on an orientable 5-manifold admits a split type, parallel tractor 3-form $\Phi \in \mathcal{E}_{[ABC]}$ compatible with the metric, that is, satisfying*

$$(\cdot \lrcorner \Phi) \wedge (\cdot \lrcorner \Phi) \wedge \Phi = \lambda g^{\mathcal{T}}(\cdot, \cdot) \operatorname{vol}^{\mathcal{T}} \tag{3.1}$$

for some $\lambda \in \mathbb{R}^*$, if and only if it is induced by a generic 2-plane field, that is, $c = c_{\mathbb{D}}$ for some generic 2-plane field $\mathbb{D} \subset TM$. (Here, $\operatorname{vol}^{\mathcal{T}} \in \Lambda^7 \mathcal{T}^*$ is the volume form of $g^{\mathcal{T}}$.)

The version of this theorem given by Hammerl and Sagerschnig, [HS09, Theorem A], restricts attention to conformal structures of signature $(2, 3)$ and instead of a parallel tractor 3-form of split type it considers a normal conformal Killing 2-form ϕ satisfying a corresponding genericity condition. However, Φ can be recovered from ϕ using the BGG splitting operator L_0 for $\Lambda^3\mathcal{T}^*$ (see (1.41)), and because any 3-form of split type on $\mathbb{R}^7$ induces a signature-$(3,4)$ symmetric bilinear form, parallel tractor 3-forms of split type only occur in signature $(2,3)$.

From this theorem and the real-analytic version of the Parallel Tractor Extension Theorem (Theorem 2.1.3), it follows immediately that the holonomy of any ambient metric induced by a real-analytic, generic 2-plane field on an orientable 5-manifold is contained in G_2, provided we restrict to the domain of the parallel extension, and with some more work that an analogous result holds for such plane fields on nonorientable 5-manifolds.

Theorem 3.1.2. *Suppose $\mathbb{D}$ is a real-analytic, generic 2-plane field on a 5-manifold, let $c_{\mathbb{D}}$ be the conformal structure it naturally induces, and let $(\widetilde{M}, \widetilde{g}_{\mathbb{D}})$ be a real-analytic ambient metric of $c_{\mathbb{D}}$. If M is orientable, then (possibly replacing $\widetilde{g}_{\mathbb{D}}$ with its restriction to some dilation-invariant open subset of $\widetilde{M}$ containing $\mathcal{G}$) $\mathrm{Hol}(\widetilde{g}_{\mathbb{D}}) \leq \mathrm{G}_2$; if M is nonorientable, then in the same sense, $\mathrm{Hol}(\widetilde{g}_{\mathbb{D}}) \leq \mathrm{G}_2 \times \mathbb{Z}_2$ but $\mathrm{Hol}(\widetilde{g}_{\mathbb{D}}) \not\leq \mathrm{G}_2$.*

Proof. First suppose that M is orientable. Let Φ be a tractor 3-form on $c_{\mathbb{D}}$ guaranteed by Theorem 3.1.1. Then Theorem 2.1.3 (the real-analytic version of the Parallel Tractor Extension Theorem) guarantees that $\widetilde{M}$ admits a parallel ambient extension $\widetilde{\Phi}$ extending Φ; because Φ is split, so is $\widetilde{\Phi}$. Then, after replacing $\widetilde{g}_{\mathbb{D}}$ with its restriction to the domain of $\widetilde{\Phi}$, $\mathrm{Hol}(\widetilde{g}_{\mathbb{D}})$ is contained in the stabilizer of a split 3-form on $\mathbb{R}^{3,4}$, namely, G_2.

If M is nonorientable, then neither is $\widetilde{M}$, and so $\mathrm{Hol}(\widetilde{M}, \widetilde{g}) \not\leq \mathrm{SO}(3,4)$; in particular, $\mathrm{Hol}(\widetilde{M}, \widetilde{g}) \not\leq \mathrm{G}_2 < \mathrm{SO}(3,4)$.

Now suppose that M is nonorientable. Let $\bar{M}$ be its orientation cover with projection $\bar{\pi} : \bar{M} \to M$, and let $\sigma : \bar{M} \to \bar{M}$ be the deck transformation of $\bar{\pi}$ that exchanges the (two) elements in each fiber of $\bar{\pi}$. Set $\bar{\mathbb{D}} := (T\bar{\pi})^{-1}(\mathbb{D}) \subset T\bar{M}$; since it is locally equivalent to $\mathbb{D}$ (via restrictions of $\bar{\pi}$), it is generic. We show that the data derived from $\mathbb{D}$ and $\bar{\mathbb{D}}$ are suitably compatible with respect to $\bar{\pi}$ and σ, and then use this to prove the claimed holonomy containment.

Since $\mathbb{D}$ and $\bar{\mathbb{D}}$ are locally equivalent, and the construction $\mathbb{D} \rightsquigarrow c_{\mathbb{D}}$ depends only on local data, the conformal structure $c_{\bar{\mathbb{D}}}$ on $\bar{M}$ induced by $\bar{\mathbb{D}}$ satisfies $c_{\bar{\mathbb{D}}} = \bar{\pi}^* c_{\mathbb{D}}$ and so $\sigma^* c_{\mathbb{D}} = c_{\mathbb{D}}$. Now, pick an arbitrary representative $g \in c_{\mathbb{D}}$; then, by construction, $\bar{g} :=$

$\bar{\pi}^* g$ is a representative of $c_{\bar{\mathbb{D}}}$. Let $\widetilde{g}_{\bar{\mathbb{D}}}$ be a real-analytic ambient metric of $c_{\bar{\mathbb{D}}}$. Then, the representatives g and $\bar{g}$ respectively induce splittings (t, x, ρ) and $(\bar{t}, \bar{x}, \bar{\rho})$ on $\widetilde{M}$ and $\widetilde{\bar{M}}$. Furthermore, $\bar{\pi}$ extends to a covering map $\widetilde{\bar{\pi}} : \widetilde{\bar{M}} \to \widetilde{M}$ defined by $\widetilde{\bar{\pi}}(\bar{t}, \bar{x}, \bar{\rho}) = (\bar{t}, \bar{\pi}(\bar{x}), \bar{\rho})$, and $\bar{\sigma}$ extends to a deck transformation $\sigma : \widetilde{\bar{M}} \to \widetilde{\bar{M}}$ defined by $\widetilde{\sigma}(\bar{t}, \bar{x}, \bar{\rho}) = (\bar{t}, \sigma(\bar{x}), \bar{\rho})$. Since the Levi-Civita connection $\widetilde{\bar{\nabla}}$ of $\widetilde{g}_{\bar{\mathbb{D}}}$ is just the pullback of the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{g}_{\mathbb{D}}$, it is preserved by $\widetilde{\sigma}$.

Again, Theorem 3.1.1 guarantees the existence of a split parallel tractor 3-form $\bar{\Phi}$ on the tractor bundle of $\bar{M}$. By construction, $\sigma^* \bar{\Phi} = -\bar{\Phi}$: σ^* preserves $\bar{\Phi}$ up to sign by symmetry, and because $\sigma^* \operatorname{vol}^{\mathcal{T}} = -\operatorname{vol}^{\mathcal{T}}$, substituting in (3.1) shows that it must reverse Φ. Then Theorem 2.1.3 guarantees the existence of a parallel ambient 3-form $\widetilde{\bar{\Phi}}$, and since the ambient construction is local, $\widetilde{\sigma}^* \widetilde{\bar{\Phi}} = -\widetilde{\bar{\Phi}}$.

For an arbitrary point $\bar{p} \in \bar{M}$, we compute the subgroup of $\mathrm{GL}(T_{\bar{p}} \bar{M})$ preserving the pair $\{\pm \widetilde{\bar{\Phi}}\}$ (setwise): First, $a.\widetilde{\bar{\Phi}}_{\bar{p}} = \widetilde{\bar{\Phi}}_{\bar{p}}$ for some $\mathrm{GL}(T_{\bar{p}} \bar{M})$ iff $a \in \mathrm{G}_2$; similarly, if $a.\widetilde{\bar{\Phi}} = -\widetilde{\bar{\Phi}}$, then $[a \cdot (-\operatorname{id})].\widetilde{\bar{\Phi}}_{\bar{p}} = \widetilde{\bar{\Phi}}_{\bar{p}}$, so $a \cdot (-\operatorname{id}) \in \mathrm{G}_2$ (and vice versa). Then, since $\mathbb{Z}_2 \cong \{\pm \operatorname{id}\}$ commutes with G_2, the subgroup preserving the pair $\{\pm \widetilde{\bar{\Phi}}_{\bar{p}}\}$ is exactly $\mathrm{G}_2 \times \mathbb{Z}_2$. Then, since $\pm \widetilde{\bar{\Phi}}$ are both parallel, the stabilizers of their restrictions in $\mathrm{GL}(T_{\bar{p}} \bar{M})$ at each respective point $\bar{p} \in \widetilde{M}$ determine a parallel $(\mathrm{G}_2 \times \mathbb{Z}_2)$-structure. Then, because $\{\pm \widetilde{\bar{\Phi}}\}$ is preserved by $\widetilde{\sigma}$, so is the $(\mathrm{G}_2 \times \mathbb{Z}_2)$-structure, which hence descends to such a structure on an open subset of $\widetilde{M}$ containing $\mathcal{G}$, to which we restrict $\widetilde{g}$. Then, by the compatibility of $\widetilde{\bar{\nabla}}$ and $\widetilde{\nabla}$, the structure is parallel with respect to the latter, and thus $\mathrm{Hol}(\widetilde{g}_D) \leq \mathrm{G}_2 \times \mathbb{Z}_2$. □

Henceforth, we may replace $\widetilde{g}_{\mathbb{D}}$ with its restriction to the domain of the parallel extension $\widetilde{\Phi}$ (which by construction is invariant under dilation) without comment.

Remark 3.1.3. One could alternately frame the proof of the above theorem in the nonorientable case by regarding a nonorientable (or even general) 2-plane field on a 5-manifold as a normal, regular parabolic geometry of type $(\mathfrak{g}_2, P')$, where P' is the stabilizer of a null ray in $\mathrm{G}_2 \times \mathbb{Z}_2$, which guarantees the existence of a reduction of the holonomy of the tractor connection to $\mathrm{G}_2 \times \mathbb{Z}_2 < \mathrm{O}(3, 4)$.

Remark 3.1.4. If M is orientable, extending (3.1) gives that the ambient metric $\widetilde{g}_{\mathbb{D}}$ and the

3-form Φ on $\widetilde{M}$ are related up to a nonzero constant factor by

$$[(\cdot \lrcorner \widetilde{\Phi}) \wedge (\cdot \lrcorner \widetilde{\Phi}) \wedge \lrcorner \widetilde{\Phi}] = \widetilde{g}_{\mathbb{D}} \widetilde{\mathrm{vol}},$$

where $\widetilde{\mathrm{vol}}$ is the volume form of $\widetilde{g}_{\mathbb{D}}$.

A converse of Theorem 3.1.2 is again a consequence of Theorem 3.1.1:

Proposition 3.1.5. *If a real-analytic ambient metric $\widetilde{g}$ of a signature $(2,3)$ real-analytic conformal 5-manifold (M,c) has holonomy contained in $\mathrm{G}_2 < \mathrm{SO}(3,4)$ (if M orientable) or $\mathrm{G}_2 \times \mathbb{Z}_2 < \mathrm{O}(3,4)$ (if M not orientable), then $c = c_{\mathbb{D}}$ for some generic 2-plane field $\mathbb{D} \subset TM$.*

To prove this, we use the following fact:

Proposition 3.1.6. *[HS09] Let $\mathbb{D}$ be a 2-plane field on an oriented 5-manifold, and let $\Phi_{ABC} \in \mathcal{E}_{ABC}$ be the corresponding parallel 3-form in Theorem 3.1.1. Its projecting part $[\Pi_0(\Phi)]_{ab} \in \mathcal{E}_{ab}[2]$ spans the line subbundle of $\Lambda^2\mathcal{T}^*$ that annihilates $\mathbb{D}^\perp$. Equivalently, because $\mathbb{D}$ is totally null, the 2-plane field defined by the (locally decomposable) weighted bivector field $[\Pi_0(\Phi)]^{ab} \in \mathcal{E}^{ab}[-2]$ is just $\mathbb{D}$ itself.*

Proof of Proposition 3.1.5. Again, first suppose that M is orientable. Since $\mathrm{Hol}(\widetilde{g}) \leq \mathrm{G}_2 < \mathrm{SO}(3,4)$, $\widetilde{g}$ admits a parallel 3-form of split type compatible with the metric; the restriction of that 3-form to a section of $\Gamma(T\widetilde{M}|_{\mathcal{G}})$ is a parallel tractor 3-form of split type compatible with the tractor metric. So, by Theorem 3.1.1 c is induced by some $\mathbb{D}$.

Now, suppose that M is nonorientable, and let $\mathcal{F}$ denote the parallel $(\mathrm{G}_2 \times \mathbb{Z}_2)$-structure on $\widetilde{g}$. Use the same notation as in the proof of the previous theorem, and furthermore denote the pullback conformal structure on $\bar{M}$ by $\bar{c} := \bar{\pi}^* c$. Since $(\widetilde{\widetilde{M}}, \widetilde{\widetilde{g}})$ is orientable, it admits a (parallel) $\mathrm{GL}_+(7,\mathbb{R})$-structure. Then, the intersection of this structure with the $(\mathrm{G}_2 \times \mathbb{Z}_2)$-structure $\widetilde{\bar{\pi}}^* \mathcal{F}$ is a parallel G_2-structure on $\widetilde{\widetilde{M}}$, $\mathrm{G}_2 < \mathrm{SO}(3,4)$ and hence $\widetilde{\widetilde{g}}$ admits a parallel 3-form $\widetilde{\widetilde{\Phi}}$ of split type compatible with $\widetilde{\widetilde{g}}$. As in the previous theorem, we may assume that $\widetilde{\sigma}^* \widetilde{\widetilde{\Phi}} = -\widetilde{\widetilde{\Phi}}$. Restricting $\widetilde{\widetilde{\Phi}}$ to the tractor 3-form bundle, $\Lambda^3 \bar{\mathcal{T}}^*$, of $\bar{c}$ yields a parallel tractor 3-form $\bar{\Phi}$ on $\bar{c}$, so by Theorem 3.1.1 $\bar{c}$ is induced by some generic 2-plane field, that is $\bar{c} = c_{\bar{\mathbb{D}}}$ for some $\bar{\mathbb{D}} \subset \Gamma(T\bar{M})$. This field is determined by the bivector field

$[\Pi_0(\bar{\Phi})]^{\sharp\sharp}$, where $\Pi : \Gamma(\Lambda^3\bar{\mathcal{T}}^*) \to \Gamma(\Lambda^2 T^*\bar{M})$ is the projection of a tractor 3-form onto its projecting part. Let $\sigma^{\mathcal{T}}$ denote the restriction $\widetilde{\sigma}|_{\mathcal{G}}$. Then, the restriction of $(\sigma^{\mathcal{T}})^*\bar{\Phi} = -\bar{\Phi}$ has projecting part $-[\Pi_0(\bar{\Phi})]$. Since $[\Pi_0(\bar{\Phi})]^{\sharp\sharp}$ and $-[\Pi_0(\bar{\Phi})]^{\sharp\sharp}$ define the same 2-plane field, $\bar{\mathbb{D}}$, that plane field is invariant under σ and so it descends to a plane field $\mathbb{D}$ on M. Since c and $\mathbb{D}$ are locally equivalent to $\bar{c}$ and $\bar{\mathbb{D}}$ and $\bar{c} = c_{\bar{\mathbb{D}}}$, we have $c = c_{\mathbb{D}}$ as desired. □

We can moreover use exploit Leistner and Nurowski's technical criterion (Lemma 1.6.15) to show that the set of real-analytic generic 2-plane fields $\mathbb{D}$ such that $\mathrm{Hol}(\widetilde{g}_{\mathbb{D}})$ has holonomy equal to G_2 is in a suitable sense dense in the set of all such (local) plane fields.

Subsection 1.6.3 (see Proposition 1.6.14 and Lemma 1.6.15) showed that to prove that a real-analytic ambient metric $\widetilde{g}_{\mathbb{D}}$ of a real-analytic, generic 2-field $\mathbb{D}$ on a 5-manifold M has holonomy equal to G_2 (and not a proper subgroup thereof), it suffices to show that (1) there is some nonempty open subset of M on which $c_{\mathbb{D}}$ is not conformally Einstein, (2) the manifold does not admit a pair (g, L) as in the statement of Lemma 1.6.15, and (3) $\widetilde{g}_{\mathbb{D}}$ is not locally symmetric.

Recall also that by Proposition 1.5.6 there is a (local) quasi-normal form for a generic 2-plane field $\mathbb{D}$: For any $s \in M$, there are local coordinates (x, y, p, q, z) mapping s to the origin and a function $F(x, y, p, q, z)$ defined on that coordinate chart such that the plane field is given there by $\mathrm{span}\{\partial_q, \partial_x + p\partial_y + q\partial_p + F\partial_z\}$. By Proposition 1.5.4 any such F (in particular defined in a neighborhood of the origin) for which F_{qq} is nonvanishing defines such a plane field $\mathbb{D}_F$. We will show that both of the conditions in Lemma 1.6.15 define pointwise algebraic conditions on the Weyl and Cotton tensors of an arbitrary representative of the induced conformal class c. Passing to the local coordinates of the quasi-normal form, these impose algebraic conditions on the space J_0^7 of 7-jets of functions F at the origin; for each condition, we will show that the set on which it is satisfied is contained in a proper algebraic subvariety of J_0^7. This will yield the following genericity result.

Theorem 3.1.7. *There is a dense subset $S \subset J_0^7$ with the following property: If F is a real-analytic function (for which F_{qq} is nonvanishing and) whose 7-jet $j_0^7(F)$ lies in S, and if $\widetilde{g}_F$ is any real-analytic ambient metric of the conformal class induced by the generic 2-plane field $\mathbb{D}_F$, then* $\mathrm{Hol}(\widetilde{g}_F) = \mathrm{G}_2$.

Theorem 3.1.2 says that the (global) holonomy of a real-analytic ambient metric $\widetilde{g}_{\mathbb{D}}$ induced by a generic 2-plane field on an oriented manifold is contained in G_2, so with the preceding theorem it shows that generically (in the sense made precise by the latter theorem), the (global) holonomy of a real-analytic ambient metric induced by a generic 2-plane field on an oriented manifold is equal to G_2.

Before giving the proof of Theorem 3.1.7, we develop the constructions involved in the mentioned algebraic conditions.

First, fix a generic 2-plane field $\mathbb{D}$ on a 5-manifold M, and pick a representative g of the induced conformal class $c_{\mathbb{D}}$. Define for any $x \in M$ the linear map $\Psi_x : T_xM \times \mathbb{R} \to \bigotimes^3 T_x^*M$ by

$$\Psi_x(X, \lambda) := W_{abcd}X^a + C_{bcd}\lambda,$$

where W and C as usual denote the Weyl and Cotton tensors of g. Recall that direct computation gives that the Weyl and Cotton tensors of the arbitrary representative $\widehat{g} = e^{2\Upsilon}g \in c_{\mathbb{D}}$ are respectively $\widehat{W} = e^{2\Upsilon}W$ and $\widehat{C}_{bcd} = C_{bcd} - W_{abcd}\Upsilon^a$. So, the image of Ψ_x, and in the particular its rank, are independent of the choice of representative. The pointwise condition corresponding to the local Einstein condition in Lemma 1.6.15 is that Ψ_x injective, that is, that $\operatorname{rank}\Psi_x = 6$.

Next, we define a condition on the Cartan curvature tensor A of $\mathbb{D}$ (see Subsection 1.5.4) that turns out to be equivalent to part of the condition involving totally null 2-planes in Lemma 1.6.15.

Definition 3.1.8. Let $\mathbb{V}$ be a vector space. A tensor $B \in S^4\mathbb{V}^*$ is **3-nondegenerate** if the only vector $Z \in \mathbb{V}$ for which $B(\cdot, Z, Z, Z) = 0$ is $Z = 0$; otherwise, B is **3-degenerate**.

Lemma 3.1.9. *Let $\mathbb{D}$ be a generic 2-plane field on a 5-manifold M, fix $x \in M$, and let $W \in (\bigotimes^4 T_x^*M)[2]$ be the Weyl tensor of the induced conformal class $c_{\mathbb{D}}$ at x. Then, A_x is 3-degenerate iff there exists a nonzero null vector $K \in T_xM$ such that $W(Y, K, K, X) = 0$ for all $Y \in T_xM$ and $X \in K^{\perp}$.*

Proof. ($\Rightarrow$) Suppose A_x is 3-degenerate, say, that the nonzero vector $Z \in \mathbb{D}_x$ satisfies $W_x(\cdot, Z, Z, Z) = 0$. Then take $K := Z$; it is nonzero by hypothesis and null because $\mathbb{D}$

is totally null. By Lemma 1.5.11, to show that $W(Y, K, K, X) = 0$, it suffices to consider just the images of arbitrary $X \in K^\perp$ and $Y \in T_xM$ under the quotient map $q_{-1} : T_xM \to T_xM/\mathbb{D}_x^\perp$. Recall from Lemma 1.5.9 that for any orientable subset $U \subset M$ there is a natural isomorphism $\tau : \mathbb{D}|_U \to (TM/\mathbb{D}^\perp)[-1]$ defined up to sign; fix a sign arbitrarily. We may write $q_{-1}(Y)|_U = \tau(V)$ for some $V \in \mathbb{D}|_U$, and by orthogonality, the image of $q_{-1}(X) = s\tau(Z|_U)$ for some constant s. Then, on U,

$$W(Y, K, K, X) = W(\tau(V), Z, Z, s\tau(Z)) = -sA(V, Z, Z, Z) = 0.$$

Since the choice of U is arbitrary, this identity holds everywhere.

($\Leftarrow$) Conversely, suppose that K satisfies $W(Y, K, K, X) = 0$ for all $X \in K^\perp$ and $Y \in T_xM$. If $K \in \mathbb{D}_x$, then set $Z := K$ and choose $X \in K^\perp$ and $Y \in T_xM$ such that $q_{-1}(X) = \tau(Z)$ and $q_{-1}(Y) = \tau(V)$; then, the computation in ($\Rightarrow$) with $s = 1$ gives $A(V, Z, Z, Z) = 0$. If $K \notin \mathbb{D}$, then because $\mathbb{D}_x^\perp - \mathbb{D}_x$ contains no null vectors, $K \in T_xM - \mathbb{D}_x^\perp$, so we may write $q_{-1}(K) = \tau(Z)$ for some (nonzero) $Z \in \mathbb{D}$. Taking $X = Z$ and letting $Y \in \mathbb{D}$ be arbitrary then gives $A(\cdot, Z, Z, Z) = 0$. □

We can now translate the conditions in Lemma 1.6.15 into pointwise conditions on Ψ and A:

Lemma 3.1.10. *Let $\mathbb{D}$ be a real-analytic, generic 2-plane field, and let $\widetilde{g}_{\mathbb{D}}$ be a real-analytic ambient metric of the induced conformal class. If there are points $x, y \in M$ such that Ψ_x is injective and A_y is 3-nondegenerate, then $\mathrm{Hol}(\widetilde{g}_{\mathbb{D}}) = \mathrm{G}_2$ if M is orientable, and $\mathrm{Hol}(\widetilde{g}_{\mathbb{D}}) = \mathrm{G}_2 \times \mathbb{Z}_2$ if M is nonorientable.*

Proof. First, assume that M is simply connected. We show that the conditions on Ψ_x and A_y in the statement are incompatible with each of the conditions in Lemma 1.6.15; then that lemma will prove the claim in that case.

First, if A is 3-nondegenerate, then it is nonzero, and thus so is W. Then, by Proposition 1.2.13 and (1.25), on $M \subset \widetilde{M}$ the ambient curvature satisfies

$$\mathbb{T}^M R_{ijkl,M} = -2R_{ijkl} = -2W_{ijkl} \neq 0.$$

So, $\widetilde{R}_{IJKL,M} \neq 0$, that is, $\widetilde{g}_{\mathbb{D}}$ is not locally symmetric.

Second, let g be an arbitrary representative of the induced conformal class $c_{\mathbb{D}}$, and suppose there is a dense subset $U \subseteq M$ such that $c_{\mathbb{D}}|_U$ admits a local Einstein representative in a neighborhood about every point, say, for an arbitrary point x, the local metric $\widehat{g} := e^{2\Upsilon} g|_V$ on the neighborhood V of x. Now, an Einstein metric is Cotton-flat: If $\widetilde{g}$ has Einstein constant λ, then $\widehat{C}_{bcd} = 2P_{b[c,d]} = 2\lambda g_{b[c,d]} = 0$. Then, $0 = (\widehat{C}_x)_{bcd} = (C_x)_{bcd} - \Upsilon^a (W_x)_{abcd} = \Psi_x(-(d\Upsilon)^\sharp, 1)$, so Ψ_x is not injective. Since x is arbitrary, and the injectivity of Ψ_x is independent of the choice of conformal representative, Ψ_x is not injective for each $x \in U$. Then, because U is dense and injectivity is an open condition, Ψ_x is injective nowhere.

Finally, if $\widetilde{g}_{\mathbb{D}}$ admits a parallel, totally null 2-plane field, then A is 3-degenerate everywhere.

Now, suppose that M is not simply connected, and consider the universal cover $(\widetilde{M}, \widetilde{\widetilde{g}}_{\mathbb{D}})$ of $(\widetilde{M}, \widetilde{g}_{\mathbb{D}})$ and arbitrary lifts $\widetilde{x}$ and $\widetilde{y}$ of x and y. As in the proof of 3.1.2, $\widetilde{\widetilde{g}}_{\mathbb{D}}$ is an ambient metric of the conformal class induced by the 2-plane field $(T\pi)^{-1}(\mathbb{D})$ on the universal cover $\bar{M}$ of M, where $\pi : \bar{M} \to M$ is the covering map. Since $\Psi_\bullet$ and $A_\bullet$ both depend only on local data, $\Psi_{\widetilde{x}}$ is injective because Ψ_x is, and $A_{\widetilde{y}}$ is 3-nondegenerate because A_y is. So, by the simply connected case, $\mathrm{Hol}(\widetilde{M}, \widetilde{\widetilde{g}}_{\mathbb{D}}) = \mathrm{G}_2$. Then, by [Pet06, Section 8.3.1], $\mathrm{Hol}(\widetilde{M}, \widetilde{\widetilde{g}}_{\mathbb{D}})$ is equal to the restricted holonomy $\mathrm{Hol}^0(\widetilde{M}, \widetilde{g}_{\mathbb{D}}) \leq \mathrm{Hol}(\widetilde{M}, \widetilde{g}_{\mathbb{D}})$, and thus Theorem 3.1.2 gives that $\mathrm{Hol}(\widetilde{M}, \widetilde{g}_{\mathbb{D}}) = \mathrm{G}_2$ if M is orientable and $\mathrm{Hol}(\widetilde{M}, \widetilde{g}_{\mathbb{D}}) = \mathrm{G}_2 \times \mathbb{Z}_2$ if not. $\square$

(Proof of Theorem 3.1.7). Recall that Nurowski's formula (A.1) gives a representative g_F of the conformal class induced by $\mathbb{D}_F$ such that $(g_F)_{ab}$ and $(g_F)^{ab}$ are both polynomial in F, derivatives of F of order no more than 4, and F_{qq}^{-1}. The Weyl and Cotton tensors of a metric are polynomial in the components of the metric, those of its inverse, and those of its derivatives of degree no more than 2 and 3, respectively, so the Weyl and Cotton tensors of g_F are polynomial in F, F_{qq}^{-1}, and the derivatives of F of order at most 6 and 7, respectively. So, to prove the claim, it suffices to show that the subsets of the jet space J_0^7 defined by the conditions on Ψ_0 and A_0 are contained in proper algebraic subvarieties of that space: Then, the complement of the union of those varieties is Zariski-open and hence dense (in the standard topology), and by Lemma 3.1.10, any function F whose 7-jet, $j_0^7(F)$, lies in that complement satisfies $\mathrm{Hol}(\widetilde{g}_F) = \mathrm{G}_2$ (provided again that we restrict to F satisfying

$F_{qq} \neq 0$).

Represent Ψ_0 as a matrix in some basis; its rank is less than 6 iff the determinants of each of its 6×6 submatrices vanish, which defines an algebraic variety. To show that it is proper, we show that its complement is nonempty. In particular, we show that Ψ_0 is injective for the functions $F[\mathbf{a}, b]$ considered by Leistner-Nurowski (see Example 1.6.13) for which $a_3 \neq 0$ and $a_4 \neq 0$. Suppose that for such a function, $(\Psi_0)_{bcd} = 0$; then, specializing to $\{bcd\} = \{415\}$ gives $0 = (\Psi_0)_{415} = W_{abcd}X^a + C_{bcd}\lambda$. Consulting the tensorial data in Appendix A.2 that Leistner and Nurowski computed for the metrics $g_{F[\mathbf{a},b]}$ gives that this condition simplifies to $A_3 X^1 = 0$. Since $a_3 \neq 0$ and $a_4 \neq 0$, $A_3 \neq 0$ and $A_4 \neq 0$, and so $X^1 = 0$. Similarly specializing $\{bcd\}$, in order, to $\{115\}$, $\{413\}$, $\{113\}$, $\{414\}$, and $\{114\}$ respectively shows that X^4, λ, X^3, X^2, and X^5 are also all zero, and so that Ψ_0 is injective as desired.

We now address the condition on A_0, which in this proof we henceforth denote just A: Specializing the formula in Subsection 1.5.2 to the origin gives that $\mathbb{D}_0 = \operatorname{span}\{\partial_q, \partial_x\}$. In the coordinates (u, v) dual to (∂_q, ∂_x), the homogeneous polynomial defined by A is

$$A(Z, Z, Z, Z) = A^{(0)}u^4 + 4A^{(1)}u^3 v + 6A^{(2)}u^2 v^2 + 4A^{(3)}uv^3 + A^{(4)}v^4$$

for some coefficients $A^{(k)} \in \mathbb{R}$, $k \in \{0, \ldots 4\}$. Now, A is 3-nondegenerate iff the equations $A(\partial_q, Z, Z, Z) = 0$ and $A(\partial_x, Z, Z, Z) = 0$ have a common nonzero solution Z. In terms of the dual basis, these conditions become

$$\begin{cases} A^{(0)}u^3 + 3A^{(1)}u^2 v + 3A^{(2)}uv^2 + A^{(3)}v^3 & = 0 \\ A^{(1)}u^3 + 3A^{(2)}u^2 v + 3A^{(3)}uv^2 + A^{(4)}v^3 & = 0. \end{cases}$$

The 5-tuple $(A^{(k)})$ of coefficients for which this system admits a nonzero (real) solution is contained in the set of 5-tuples for which the system admits a solution $(u, v) \in \mathbb{C}^2 - \{(0, 0)\}$, which is in turn characterized by the vanishing of the resultant of the system:

$$\det \begin{pmatrix} A^{(0)} & 3A^{(1)} & 3A^{(2)} & A^{(3)} & 0 & 0 \\ 0 & A^{(0)} & 3A^{(1)} & 3A^{(2)} & A^{(3)} & 0 \\ 0 & 0 & A^{(0)} & 3A^{(1)} & 3A^{(2)} & A^{(3)} \\ A^{(1)} & 3A^{(2)} & 3A^{(3)} & A^{(4)} & 0 & 0 \\ 0 & A^{(1)} & 3A^{(2)} & 3A^{(3)} & A^{(4)} & 0 \\ 0 & 0 & A^{(1)} & 3A^{(2)} & 3A^{(3)} & A^{(4)} \end{pmatrix} = 0. \tag{3.2}$$

We show that the map $J^7_0 \to S^4\mathbb{D}^*_0$ sending a 7-jet of $\mathbb{D}$ (equivalently, F) to its symmetric 4-form A is surjective by perturbing the flat model, $F(x, y, p, q, z) = q^2$, at a high order. Then, since (3.2) is nontrivial on $S^4\mathbb{D}^*_0$, that equation defines a proper subvariety of J^7_0.

We thus henceforth specialize to functions $F(x, y, p, q, z) = q^2 + f$, where f vanishes to order 6. (If a function h vanishes to order m in those variables together, denote $h = 0 \mod O(|\mathbf{x}|^m)$.) We first compute for such functions the representative $g_F \in c_F$ given by Nurowski's formula (A.1), up to second order. Recall that D denotes the first-order differential operator $\partial_x + p\partial_y + q\partial_p + F\partial_q$; precomputing the derivatives of F that appear in (A.1) gives that

$$\left.\begin{aligned} F_q &= 2q \\ F_{qq} &= 2 \\ F_{qqqq} &= f_{qqqq} \\ DF_{qqq} &= f_{qqqx} \\ D^2F_{qq} &= f_{qqxx} \end{aligned}\right\} \quad \mod O(|\mathbf{x}|^3),$$

and that all other derivatives of F that occur vanish modulo $O(|\mathbf{x}|^3)$. Substituting in (A.1) leaves just

$$\begin{aligned} g_F = -3(D^2F_{qq})F^3_{qq}(\widetilde{\omega}^1)^2 + 6(DF_{qqq})F^2_{qq}\widetilde{\omega}^1\widetilde{\omega}^2 + 30F^4_{qq}\widetilde{\omega}^1\widetilde{\omega}^4 \\ - 3F_{qq}F_{qqqq}(\widetilde{\omega}^2)^2 + 30F^3_{qq}\widetilde{\omega}^2\widetilde{\omega}^5 - 20F^4_{qq}(\widetilde{\omega}^3)^2 + O(|\mathbf{x}|^3), \end{aligned}$$

where the coframe $(\widetilde{\omega}^a)$ is given in Appendix A.1. Substituting for the $\widetilde{\omega}^a$ (and again discarding the terms that vanish modulo $O(|\mathbf{x}|^3)$) gives

$$\begin{aligned} g_F = &-80q^2\,dx^2 + 160q\,dx\,dp - 480p\,dx\,dq + 240\,dx\,dz \\ &- 24f_{qqxx}\,dy^2 + 480\,dy\,dq + 24f_{qqqx}\,dy\,dz - 320\,dp^2 - 6f_{qqqq}\,dz^2 + O(|\mathbf{x}|^3). \end{aligned}$$

In order to compute A for this F, we compute explicitly the isomorphism $\tau_0 : \mathbb{D}_0 \to (T_0M/\mathbb{D}_0^\perp)[-1]$ (up to overall scale); trivializing the codomain using g_F defines an isomorphism $\mathbb{D}_0 \to T_0M/\mathbb{D}_0^\perp$, which we also denote τ_0. Up to a constant factor, the g_F-skew trivialized isomorphism μ_0 is given by

$$\mu_0 : \begin{cases} \partial_q \mapsto -\,dx \\ \partial_x \mapsto \quad dq \end{cases}.$$

Raising indices gives that the trivialized isomorphism ψ_0 is given up to a constant factor by

$$\psi_0 : \begin{cases} dx \mapsto 2\partial_z + \mathbb{D}^\perp \\ dq \mapsto \partial_y + \mathbb{D}^\perp \end{cases}.$$

Composing then gives that, again up to a constant factor, τ_0 is given by

$$\tau_0 : \begin{cases} \partial_q \mapsto -2\partial_z + \mathbb{D}^\perp \\ \partial_x \mapsto \quad\ \partial_y + \mathbb{D}^\perp \end{cases}.$$

So, using the definition of A (Subsection 1.5.4) gives again up to an overall constant factor that

$$\begin{aligned} A^{(0)} &= 4W(\partial_q, \partial_z, \partial_q, \partial_z) \\ A^{(1)} &= -2W(\partial_q, \partial_z, \partial_q, \partial_y) \\ A^{(2)} &= -2W(\partial_q, \partial_z, \partial_x, \partial_y) \\ A^{(3)} &= W(\partial_q, \partial_y, \partial_x, \partial_y) \\ A^{(4)} &= W(\partial_x, \partial_y, \partial_x, \partial_y). \end{aligned}$$

Computing W (at 0) in coordinates using the previous formula for g_F modulo $O(|\mathbf{x}|^3)$ gives up to some (overall) constant factor that

$$\begin{aligned}
A^{(0)} &= f_{qqqqqq}(0) \\
A^{(1)} &= f_{qqqqqx}(0) \\
A^{(2)} &= f_{qqqqxx}(0) \\
A^{(3)} &= f_{qqqxxx}(0) \\
A^{(4)} &= f_{qqxxxx}(0).
\end{aligned} \tag{3.3}$$

For f as above, we may thus identify A with $(\nabla^4 f_{qq})(0)|_{\mathbb{D}_0}$, where $(\nabla^4 f_{qq})(0)$ denotes the symmetric 4-tensor defined by the coefficients of the order 4 Taylor polynomial of f_{qq} at 0, and $\cdot|_{\mathbb{D}_0}$ denotes restriction to $\mathbb{D}_0$. Since we may prescribe those coefficients freely when choosing f, (3.3) shows that the map $J^7_0 \to S^4\mathbb{D}^*_0$ is surjective as desired, completing the proof. □

Remark 3.1.11. Because of the explicit computational nature of the conditions, it is easy (albeit tedious) to generate large new classes of metrics with holonomy equal to G_2.

3.1.2 Einstein Nurowski conformal structures

If $\mathbb{D}$ is a real-analytic field such that $c_{\mathbb{D}}$ contains an Einstein representative g, then a real-analytic ambient metric $\widetilde{g}_{\mathbb{D}}$ inherits extra geometric structure, including a natural foliation by hypersurfaces which in turn admit structures depending on the sign of the Einstein constant of g. We first recall definitions of some of the structures that occur in this setting.

Definition 3.1.12. An **almost complex structure** on a manifold N is a bundle map $\mathbb{J} : TN \to TN$ satisfying $\mathbb{J}^2 = -\operatorname{id}_{TN}$. An almost complex structure $\mathbb{J}$ is **compatible** with a given pseudo-Riemannian metric γ on N if $\gamma(\mathbb{J}A, \mathbb{J}B) = \gamma(A, B)$, that is, if $\mathbb{J}_{ab}$ is antisymmetric; in that case, $h_{ab} := \gamma_{ab} - i\mathbb{J}_{ab}$ is a hermitian metric. If moreover $\nabla\mathbb{J} = 0$, then there is a (unique) complex structure on N suitably compatible with $\mathbb{J}$. In this case, the pair $(\gamma, \mathbb{J})$ is called a **Kähler structure** on N, and $(N, \gamma, \mathbb{J})$ is called a **Kähler manifold**.

In particular, a Kähler manifold $(N, \gamma, \mathbb{J})$ of signature, say, (p, q), is simultaneously a pseudo-Riemannian manifold (with metric γ_{ab}), a complex manifold with pseudo-Hermitian metric h_{ab}, and a symplectic manifold (with symplectic form $\mathbb{J}_{ab}$), and these structures are compatible in the sense that the sense that $\mathbb{J}$ is compatible with γ and h can be recovered from those two objects using the given formula. (In fact, any of these three objects can be recovered from the other two.) One can show that if $(N, \gamma, \mathbb{J})$ is Kähler, then p and q must both be even. If (N, γ) admits a Kähler structure defined by a parallel almost complex structure $\mathbb{J}$, then $\mathrm{Hol}(\gamma)$ is contained in the stabilizer of $\mathbb{J}_x$ in $\mathrm{O}(\gamma_x) = \mathrm{O}(p, q)$, namely, $\mathrm{U}(\frac{p}{2}, \frac{q}{2})$.

Changing some signs yields analogous structures.

Definition 3.1.13. An **almost paracomplex structure** on a manifold N is a bundle map $\mathbb{K} : TN \to TN$ satisfying $\mathbb{K}^2 = \mathrm{id}_{TN}$ such that the ± 1-eigenbundles have the same rank. An almost complex structure $\mathbb{K}$ is **compatible** with a given pseudo-Riemannian metric γ on N if $\gamma(\mathbb{K}A, \mathbb{K}B) = -\gamma(A, B)$, that is, if $\mathbb{K}_{ab}$ is antisymmetric. If moreover $\nabla \mathbb{K} = 0$, then $\mathbb{K}$ arises from some paracomplex structure on TN (a paracomplex structure is an analog of a complex structure—see [CFG96]), the pair $(\gamma, \mathbb{J})$ is called a **para-Kähler structure** on N, and $(N, \gamma, \mathbb{J})$ is called a **para-Kähler manifold**.

A para-Kähler manifold is simultaneously a pseudo-Riemannian manifold, a paracomplex manifold, and a symplectic manifold, and those structures are compatible in a sense analogous to that in the discussion of Kähler structures above. If $(N, \gamma, \mathbb{K})$ is a para-Kähler manifold, then necessarily its dimension is even, say, $2m$, and γ has neutral signature (m, m). If (N, γ) admits a para-Kähler structure defined by a parallel almost paracomplex structure $\mathbb{K}$, then $\mathrm{Hol}(\gamma)$ is contained inside the stabilizer of $\mathbb{K}_x$ in $\mathrm{O}(\gamma_x) = \mathrm{O}(m, m)$, namely, $\mathrm{GL}(m, \mathbb{R})$.

We can specialize Proposition 2.1.14 about the ambient metrics of Einstein conformal structures to the case of such structures induced by real-analytic, generic 2-plane fields $\mathbb{D}$.

Proposition 3.1.14. *Let $\mathbb{D}$ be a real-analytic, generic 2-plane field on an orientable 5-manifold whose induced conformal structure $c_{\mathbb{D}}$ admits an Einstein representative with Einstein constant λ, and let $\Xi \in \Gamma(T\widetilde{M})$ be the corresponding parallel ambient vector field (see Proposition 2.1.14).*

- *If* $\lambda < 0$*, then* Ξ *is spacelike,* $\mathrm{Hol}(\widetilde{g}_{\mathbb{D}}^{\mathrm{can}}) \leq \mathrm{SU}(1,2)$*, and* $\widetilde{M}$ *admits a natural foliation by* 6*-dimensional leaves, each of which admits a natural signature-*$(2,4)$ *Kähler structure compatible with the pullback metric.*

- *If* $\lambda = 0$ *(i.e., g is Ricci-flat), then* Ξ *is lightlike, and* $\mathrm{Hol}(\widetilde{g}_{\mathbb{D}}^{\mathrm{can}}) \leq \mathrm{SL}(2,\mathbb{R}) \ltimes N$*, where* N *is the distinguished nilpotent subgroup* (1.52) *of the parabolic subgroup* $P < \mathrm{G}_2$ *(this is just the subgroup of* G_2 *preserving a null vector). Also,* $\widetilde{M}$ *admits a parallel partial flag structure of type* $(1,3,4,6)$*, which hence defines nested foliations whose leaves have dimensions* 1*,* 3*,* 4*, and* 6*; the* 1*- and* 3*-dimensional leaves are totally isotropic submanifolds, and the* 4*- and* 6*-dimensional leaves are coisotropic. Furthermore,* M *admits natural nested foliations by totally isotropic* 2*-dimensional leaves and coisotropic* 3*-dimensional leaves.*

- *If* $\lambda > 0$*, then* Ξ *is timelike,* $\mathrm{Hol}(\widetilde{g}_{\mathbb{D}}^{\mathrm{can}}) \leq \mathrm{SL}(3,\mathbb{R})$*, and* $\widetilde{M}$ *admits a natural foliation by* 6*-dimensional leaves, each of which admits a natural para-Kähler structure compatible with the pullback metric.*

Also, if $\lambda \neq 0$*, then* $\widetilde{g}_{\mathbb{D}}^{\mathrm{can}}$ *can be locally realized as a product metric.*

Leistner and Nurowski allude to the holonomy containments in the cases for which $\lambda \neq 0$ [LN10, Remark 5].

Proof. The relationship between the sign of λ and $\widetilde{g}_{\mathbb{D}}^{\mathrm{can}}(\Xi,\Xi)$ and the result about the local product decomposition were shown in 2.1.14.

Let $[\Xi]$ denote the line spanned by Ξ, and let $\widetilde{\Phi} \in \Gamma(\Lambda^3 T^* \widetilde{M})$ denote the parallel ambient 3-form induced by $\mathbb{D}$ as in the proof of Theorem 3.1.2. Since $[\Xi]$ is parallel, so is the 6-plane field $[\Xi]^\perp$, and thus it defines a foliation $\mathcal{F}$ of $\widetilde{M}$ by 6-dimensional leaves. For any leaf $L \in \mathcal{F}$, let ι_L denote the inclusion $L \hookrightarrow \widetilde{M}$.

We know from Theorem 3.1.2 that $\mathrm{Hol}(\widetilde{g}_{\mathbb{D}}^{\mathrm{can}}) \leq \mathrm{G}_2$. So, since Ξ is also parallel, $\mathrm{Hol}(\widetilde{g}_{\mathbb{D}}^{\mathrm{can}})$ is contained in the stabilizer in G_2 of an arbitrary nonzero vector $X \in \mathbb{R}^{3,4}$ such that $\langle X, X\rangle$ has the same sign as $\widetilde{g}_{\mathbb{D}}^{\mathrm{can}}(\Xi,\Xi) = -2\lambda$. If $\lambda < 0$, this is just the stabilizer in G_2 of a point in the sphere $\mathbb{S}^{2,4}$, which one can compute is $\mathrm{SU}(1,2)$. If $\lambda = 0$, this is just the stabilizer in

G_2 of a null vector, which is $\mathrm{SL}(2,\mathbb{R}) \rtimes N$. If $\lambda > 0$, this is the stabilizer in G_2 of a point in the sphere $\mathbb{S}^{3,3}$, which is $\mathrm{SL}(3,\mathbb{R})$.

Suppose now that $\lambda < 0$ and consider an arbitrary leaf $L \in \mathcal{F}$. Since $\widetilde{g}^{\mathrm{can}}_{\mathbb{D}}(\Xi, \Xi) > 0$, the pullback $\iota_L^* \widetilde{g}^{\mathrm{can}}_{\mathbb{D}}$ is a signature-$(2,4)$ metric. Now, because both Ξ and $\widetilde{\Phi}$ are parallel, the 2-form $\Xi \lrcorner \widetilde{\Phi}$ is parallel, and because L is totally geodesic (since it is a leaf of a foliation of a parallel plane field), the skew endomorphism $\mathbb{J} := [\iota_L^*(\Xi \lrcorner \widetilde{\Phi})]^\sharp \in \Gamma(\mathrm{End}_{\mathrm{skew}}(TL))$ is parallel with respect to $\iota_L^* \widetilde{g}^{\mathrm{can}}_{\mathbb{D}}$. Computing an example in $\operatorname{Im} \widetilde{\mathbb{O}}$ and using the transitivity of the action of G_2 on $\mathbb{S}^{2,4} \subset \operatorname{Im} \widetilde{\mathbb{O}}$ shows that $\mathbb{J}^2 = -\operatorname{id}$, so the pair $(\iota_L^* \widetilde{g}^{\mathrm{can}}_{\mathbb{D}}, \mathbb{K})$ defines an almost Hermitian structure on L; since $\mathbb{J}$ is parallel it in fact defines a signature-$(2,4)$ Kähler structure on L.

The case $\lambda > 0$ is similar. Since $\widetilde{g}^{\mathrm{can}}_{\mathbb{D}}(\Xi, \Xi) > 0$, $\iota_L^* \widetilde{g}^{\mathrm{can}}_{\mathbb{D}}$ is a signature-$(3,3)$ metric. Now, the skew endomorphism $\mathbb{K}$ defined by the same formula as for $\mathbb{J}$ in the case $\lambda < 0$ satisfies $\mathbb{K}^2 = \operatorname{id}$ and the ± 1-eigenspaces of $\mathbb{K}$ at any (equivalently, every) point in p both have dimension 3, so the same reasoning as in the previous case shows that $(\iota_L^* \widetilde{g}^{\mathrm{can}}_{\mathbb{D}}, \omega)$ defines a para-Kähler structure on L.

Finally suppose that $\lambda = 0$. Then, the equation $\widetilde{g}^{\mathrm{can}}_{\mathbb{D}}([\Xi], [\Xi]) = 0$ implies that $[\Xi] \subset [\Xi]^\perp$, and so the foliation by integral curves of $[\Xi]$ is nested inside the foliation $\mathcal{F}$ of $\widetilde{M}$ defined by $[\Xi]^\perp$. Computing an example in $\operatorname{Im} \widetilde{\mathbb{O}}$ and using the transitivity of G_2 on the null cone $\mathcal{N} \subset \operatorname{Im} \widetilde{\mathbb{O}}$ shows that $(\Xi \lrcorner \widetilde{\Phi}) \wedge (\Xi \lrcorner \widetilde{\Phi}) \wedge (\Xi \lrcorner \widetilde{\Phi}) = 0$, so the pullback of $\Xi \lrcorner \widetilde{\Phi}$ to a leaf of $\mathcal{F}$ does not define a symplectic structure as it does when $\lambda \neq 0$. There is, however, a substitute structure for this case: Checking again an example in the flat model shows that $(\Xi \lrcorner \widetilde{\Phi}) \wedge (\Xi \lrcorner \widetilde{\Phi})$ is decomposable and null, so define the trivector field

$$\Delta := [\widetilde{*}[(\Xi \lrcorner \widetilde{\Phi}) \wedge (\Xi \lrcorner \widetilde{\Phi})]]^{\sharp\sharp\sharp} \in \Gamma(\Lambda^3 T\widetilde{M}).$$

(Here, $\sharp\sharp\sharp$ just indicates raising of all three indices.) Since Ξ and $\widetilde{\Phi}$ are parallel, and $\widetilde{*}$ commutes with covariant differentiation, Δ too is parallel, and so it defines a parallel, null 3-plane field $[\Delta]$. Checking an example shows that $[\Xi] \subset [\Delta] \subset [\Delta]^\perp$, yielding a refined filtration,

$$[\Xi] \subset [\Delta] \subset [\Delta]^\perp \subset [\Xi]^\perp,$$

which we may view as a bundle of incomplete flags of null and conull subspaces. Each plane field in the filtration is parallel, so it defines nested foliations of $\widetilde{M}$ by leaves of dimensions 1, 3, 4, and 6.

To see the claim about M it is enough to consider the restrictions of the above data to $\mathcal{G}$. Since $\lambda = 0$, computing directly using Example 2.1.8 gives that $\Xi = t^{-1}\partial_\rho$, and so $[\Xi]|_{\mathcal{G}} = \operatorname{span}\{\partial_\rho\}|_{\mathcal{G}}$. So, each leaf is transverse to $\mathcal{G}$, and so the intersections of the leaves in the nested foliations of $\widetilde{M}$ with $\mathcal{G}$ comprise nested foliations of $\mathcal{G}$ by leaves of dimensions 2, 3, and 5. The normal form of $\widetilde{g}_{\mathbb{D}}^{\mathrm{can}}$ shows that, with respect to the decomposition realizing $\widetilde{M}$ as a subset of $\mathbb{R}^+ \times M \times \mathbb{R}$, we have $[\Xi]^\perp = TM \oplus \operatorname{span}\{\partial_\rho\}$, the 6-dimensional plane field $[\Xi]^\perp|_{\mathcal{G}} \subset T\widetilde{M}|_{\mathcal{G}}$ is just $TM|_{\mathcal{G}} \oplus \operatorname{span}\{\partial_\rho\}|_{\mathcal{G}}$. So the 5-dimensional plane field is just $TM|_{\mathcal{G}} \subset T\mathcal{G}$, and the leaves of the corresponding foliation are just the submanifolds $\{t = t_0\} \subset \mathcal{G}$; regarded as sections of $\mathcal{G} \to M$, these are, respectively, just the Ricci-flat representatives $t_0^2 g$ of c homothetic to g. By homogeneity, the projections of the foliations of $M \times \{t_0\}$ to M by 2- and 3-dimensional leaves by the projection $M \times \{t_0\}$ do not depend on t_0, and so they determine foliations of M by 2- and 3-dimensional leaves naturally associated to the Ricci-flat representative $g \in c$. Since $[\Delta]|_{\mathcal{G}}$ is null, consulting again the normal form of the ambient metric gives that the 2-dimensional leaves are totally null. □

Remark 3.1.15. If $\lambda \neq 0$, we can use the Kähler or para-Kähler structures on the leaves to recover local versions of the holonomy containments asserted in the proposition. Since $\widetilde{g}_{\mathbb{D}}^{\mathrm{can}}(\Xi, \Xi) \neq 0$, we have $[\Xi]^\perp \cap [\Xi] = \{0\}$, so the de Rham theorem guarantees that near an arbitrary point $p \in \widetilde{M}$, some neighborhood $(U, \widetilde{g}_{\mathbb{D}}^{\mathrm{can}}|_U)$ of p in $(\widetilde{M}, \widetilde{g}_{\mathbb{D}}^{\mathrm{can}})$ can be realized as a product $(N, g_6) \times (J, \pm ds^2)$, where N can be identified with an open set in the leaf L of $\mathcal{F}$ through p and g_6 can be identified with pullback $\iota_L^* \widetilde{g}_{\mathbb{D}}^{\mathrm{can}}$ to that set, J is an interval, and $\pm$ is the opposite of the sign of λ. In particular,

$$\operatorname{Hol}(\widetilde{g}_{\mathbb{D}}^{\mathrm{can}}|_U) \cong \operatorname{Hol}(g_6) \times \operatorname{Hol}(\pm ds^2) \cong \operatorname{Hol}(g_6).$$

Remark 3.1.16. It can be shown efficiently, independently of the arguments in the above proof that the metrics on the 6-dimensional leaves L admit Kähler or para-Kähler structures, that the (parallel) 2-forms $\iota_L^*(\Xi \lrcorner \widetilde{\Phi})$ define symplectic structures when $\lambda \neq 0$: By Remark

3.1.4, up to a nonzero constant factor we have

$$(\Xi \lrcorner \widetilde{\Phi}) \wedge (\Xi \lrcorner \widetilde{\Phi}) \wedge \widetilde{\Phi} = \widetilde{g}^{\mathrm{can}}_{\mathbb{D}}(\Xi, \Xi)\, \widetilde{\mathrm{vol}} = -2\lambda\, \widetilde{\mathrm{vol}},$$

where $\widetilde{\mathrm{vol}}$ is the volume form of $\widetilde{g}^{\mathrm{can}}_{\mathbb{D}}$. Then, by definition of $\mathbb{J}$ (and lowering an index and regarding it as a 2-form), contracting Ξ into both sides and pulling back to L gives that $\mathbb{J} \wedge \mathbb{J} \wedge \mathbb{J}$ is a nonzero multiple of the volume form of the metric on the leaf, and likewise for $\mathbb{K}$.

3.2 Integrability conditions for parallel tractor tensors

The Parallel Tractor Extension Theorem can be used to generate a large number of integrability conditions for the existence of parallel tractor tensors. The discussion in this section generalizes a similar observation about standard tractors in [Gov, §3] which uses just the tractor curvature instead of the full ambient curvature, and some results of [Lei04] for tractor forms. In the notation of this section, the observation there covered the case in which $r = 1$ and $ABC_1 \cdots C_s = abc_1 \cdots c_s$.

3.2.1 Odd dimension

First suppose n odd. Fix a parallel tractor tensor $\chi \in \Gamma(\bigotimes^r \mathcal{T}^*)$; the theorem guarantees the existence of an ambient extension $\widetilde{\chi} \in \Gamma(\bigotimes^r T^*\widetilde{M})$ of χ satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^\infty)$. Commuting covariant derivatives gives $\widetilde{R}.\widetilde{\chi} = O(\rho^\infty)$. Here, we regard $\widetilde{R}$ as a section of $\Lambda^2 T^*\widetilde{M} \otimes \mathrm{End}(T\widetilde{M})$, and . denotes the (tensorial) bundle map induced by the natural action of $\mathrm{End}(T\widetilde{M})$ on $\bigotimes^r T^*\widetilde{M}$; explicitly:

$$(\widetilde{R}.\widetilde{\chi})_{AB\mathbf{I}} = -\sum_{u=1}^{r} \widetilde{R}_{AB}{}^{Q}{}_{I_u} \widetilde{\chi}_{I_1 \cdots I_{u-1} Q I_{u+1} \cdots I_r}. \tag{3.4}$$

Restricting to $\mathcal{G}$ gives

$$[(\widetilde{R}|_{\mathcal{G}}).\chi]_{AB\mathbf{I}} = 0, \tag{3.5}$$

where . now denotes the restriction of the above action to sections restricted to $\mathcal{G}$. We can regard $\widetilde{R}|_{\mathcal{G}}$ as a tractor tensor and thus view (3.5) as a condition on the tractor χ. Specializing the free multi-index AB realizes that condition more concretely. For example, taking A and B to be tangent to $\mathcal{G}$ recovers the fact that the tractor curvature annihilates the tractor tensor χ. At least for tractor r-forms, the remaining nontrivial choice up to symmetry, $AB = \infty b$, just yields a condition that is a differential consequence of that annihilation. (Taking either index to be 0 makes the left-hand side zero.)

If we furthermore fix a representative $g \in c$, specializing $\mathbf{I}$ according to the induced splitting just realizes the integrability conditions as tensorial conditions on the tractor components; specializing to tractor r-forms recovers formulas of Leitner [Leitner]. Equation (6.2) of [FG] gives the ambient curvature along $\mathcal{G}$ can be realized explicitly in terms of various tensors associated to (M, g):

$$\begin{aligned} \widetilde{R}_{ijkl}|_M &= W_{ijkl} \\ \widetilde{R}_{\infty jkl}|_M &= C_{jkl} \\ \widetilde{R}_{\infty jk\infty}|_M &= -(n-4)^{-1} B_{jk}; \end{aligned}$$

here, W, C, and B are respectively the Weyl, Cotton, and Bach tensors of g. So, for example, if χ is a tractor r-form, with components $(\chi_-)_{i_2 \cdots i_r}$, $(\chi_0)_{i_1 \cdots i_r}$, $(\chi_\mp)_{i_3 \cdots i_r}$, and $(\chi_+)_{i_2 \cdots i_r}$ as in (2.1), the condition determined above by $AB = ab$ becomes

$$\begin{cases} (W.\chi_-)_{abi_2\cdots i_r} &= 0 \\ (W.\chi_0)_{abi_1\cdots i_r} &= -rC_{[i_1|ab|}(\chi_-)_{i_2\cdots i_r]} \\ (W.\chi_\mp)_{abi_3\cdots i_r} &= C^q{}_{ab}(\chi_-)_{qi_2\cdots i_r} \\ (W.\chi_+)_{abi_2\cdots i_r} &= C^q{}_{ab}(\chi_0)_{qi_2\cdots i_r} + (r-1)C_{[i_2|ab|}(\chi_\mp)_{i_3\cdots i_r]} \end{cases}, \tag{3.6}$$

where by raising an index we regard W as a section of $\bigotimes^2 T^*M \otimes \operatorname{End}(TM)$. Here, . is the tensorial bundle map

$$\Gamma(\textstyle\bigotimes^d T^*M \otimes \operatorname{End}(TM)) \times \Gamma(\bigotimes^r T^*M) \to \Gamma(\bigotimes^d T^*M \otimes \bigotimes^r T^*M)$$

induced by the natural action of $\operatorname{End}(TM)$ on $\bigotimes^r T^*M$ analogous to the map . defined in (3.4): More generally, define the map . for simple tensor products $\mu \otimes \varphi \in \Gamma(\bigotimes^d T^*M \otimes \operatorname{End}(TM))$ and $\beta \in \Gamma(\bigotimes^r T^*M)$ by $(\mu \otimes \varphi).\beta = \mu \otimes (\varphi \cdot \beta)$, where $\cdot$ denotes the natural action of $\Gamma(\operatorname{End}(TM))$ on $\Gamma(\bigotimes^r T^*M)$, namely,

$$(\varphi \cdot \beta)_{i_1 \cdots i_r} = -\sum_{u=1}^{r} \varphi^q{}_{u_i} \beta_{i_1 \cdots i_{u-1} q i_{u+1} \cdots i_r}.$$

Unwinding the definitions then gives, explicitly,

$$(W.\beta)_{abj_1\cdots j_l} = -\sum_{u=1}^{l} W_{ab}{}^{q}{}_{j_u}\beta_{j_1\cdots j_{u-1}qj_{u+1}\cdots j_l}.$$

(If $r = 1$, in which case a tractor form has no component $\chi_{\mp}$, the above condition involving $W.\chi_{\mp}$ disappears, as does the term involving $\chi_{\mp}$ on the right-hand side of the last equation.) These conditions can be used to recover some facts about structures encoded by parallel tractors. For example, if (M, c) is almost Einstein, let U be the (dense) subset of M on which the Einstein scale is nonsingular. In the splitting induced by the Einstein representative of $c|_U$, the corresponding parallel tractor form on U is $(1, 0, -\lambda)$. Then, the third equation gives that $C|_U = 0$, and thus by continuity $C = 0$, recovering the fact that almost Einstein conformal structures are conformally Cotton-flat.

For tractor tensors of irreducible type, we may combine the conditions produced as above with the BGG splitting operator for that type to produce integrability conditions for a parallel tractor tensor χ in terms of just its projecting parts α (which, recall, for irreducible type determines the full tractor tensor). Again for forms, this yields the conditions:

$$\begin{cases} (W.\alpha)_{abi_2\cdots i_r} &= 0 \\ (W.d\alpha)_{abi_1\cdots i_r} &= -rC_{[i_1|ab|}\alpha_{i_2\cdots i_r]} \\ (W.d^*\alpha)_{abi_3\cdots i_r} &= (n-r+2)C^{q}{}_{ab}\alpha_{qi_2\cdots i_r} \\ (W.\Box\alpha)_{abi_2\cdots i_r} &= \frac{1}{r}C^{q}{}_{ab}(d\alpha)_{qi_2\cdots i_r} + \frac{r-1}{n-r+2}C_{[i_2|ab|}(d^*\alpha)_{i_3\cdots i_r]} \end{cases},$$

where $\Box$ is the operator in [Lei04, Section 4].

Similarly, the condition determined by $AB = \infty b$ becomes

$$\begin{cases} (C.\chi_-)_{bi_2\cdots i_r} &= 0 \\ (C.\chi_0)_{bi_1\cdots i_r} &= -(n-4)^{-1}rB_{b[i_1}(\chi_-)_{i_2\cdots i_r]} \\ (C.\chi_{\mp})_{bi_3\cdots i_r} &= (n-4)^{-1}B_b{}^{q}(\chi_-)_{qi_3\cdots i_r} \\ (C.\chi_+)_{bi_2\cdots i_r} &= (n-4)^{-1}[B_b{}^{q}(\chi_+)_{qi_2\cdots i_r} + (r-1)B_{b[i_2}(\chi_{\mp})_{i_3\cdots i_r]}] \end{cases}. \tag{3.7}$$

Here, . is the tensorial action of C on $\Gamma(\bigotimes^l T^*M)$ analogous to the above action for W: Raising an index allows us to regard C as a section of $T^*M \otimes \mathrm{End}(TM)$. The action . is again the one induced by that of $\Gamma(\mathrm{End}(TM))$ on covariant tensors:

$$(C.\beta)_{bj_1\cdots j_l} = -\sum_{u=1}^{l} C_b{}^{q}{}_{j_u}\beta_{j_1\cdots j_{i-1}qj_{u+1}\cdots j_l}.$$

The above systems of integrability conditions were produced using a somewhat different method in [Leitner, Section 5].

Remark 3.2.1. Computing directly shows that the conditions (3.6) can be written compactly using the bundle map F_χ defined at the beginning of Subsection 2.2.1, as

$$F_\chi(C(X,Y)) = W(X,Y).\chi,$$

where X and Y are arbitrary sections of TM, W is regarded as a $(3,1)$-tensor, so that $W(X,Y) \in \mathrm{End}(TM)$, and . denotes the action of $\mathrm{End}(TM)$ on $\Lambda^r \mathcal{T}^*$ defined by acting separately on its components in $\Lambda^* T^* M$ determined by the splitting with respect to the representative $g \in c$.

Similarly, we may write the system (3.7) as

$$F_\chi(B(X)) = (4-n)C(X).\chi,$$

where X is an arbitrary section of TM, and C is regarded as a $(2,1)$-tensor, so that $C(X) \in \mathrm{End}(TM)$.

Now, differentiating $\widetilde{R}.\widetilde{\chi}$ (see (3.4)) arbitrarily many, say, s, times and then restricting to $\rho = 0$ gives $[(\widetilde{\nabla}^s \widetilde{R})|_{\mathcal{G}}].\chi = 0$, where . now denotes the restriction of the above action to sections restricted to $\mathcal{G}$. Now, we can regard $(\widetilde{\nabla}^s \widetilde{R})|_{\mathcal{G}}$ as a section of a section of a suitably weighted tractor tensor bundle [FG, Proposition 6.5], and thus again regard

$$[(\widetilde{\nabla}^s \widetilde{R})|_{\mathcal{G}}].\chi = 0$$

as a strictly tractor equation.

As above, we can produce systems of integrability conditions by specializing the free indices on $\widetilde{R}$ and can furthermore explicitly write the conditions in terms of tractor components in a splitting induced by a choice of representative $g \in c$. For example, when $s = 1$, in the special case of tractor forms, specializing all the free indices to lie tangent to the factor M in the splitting $\widetilde{M} \leftrightarrow \mathbb{R}^+ \times M \times \mathbb{R}$ induced by a choice $g \in c$ gives (consulting equation

(6.3) in [FG]) the conditions

$$\begin{cases} (V.\chi_-)_{abci_2\cdots i_r} &= 0 \\ (V.\chi_0)_{abci_1\cdots i_r} &= -rY_{[i_1|abc|}(\chi_-)_{i_2\cdots i_r]} \\ (V.\chi_\mp)_{abci_3\cdots i_r} &= Y^q{}_{abc}(\chi_-)_{qi_2\cdots i_r} \\ (V.\chi_+)_{abci_2\cdots i_r} &= Y^q{}_{abc}(\chi_0)_{qi_2\cdots i_r} + (r-1)Y_{[i_2|ab|}(\chi_\mp)_{i_3\cdots i_r]} \end{cases},$$

where

$$V_{ijklm} = W_{ijkl,m} + g_{im}C_{jkl} - g_{jm}C_{ikl} + g_{km}C_{lij} - g_{lm}C_{kij}$$
$$Y_{jklm} = C_{jkl,m} - P_m{}^iW_{ijkl} + (n-4)^{-1}(g_{km}B_{jl} - g_{lm}B_{jk}).$$

Raising the k indices, we may regard V and Y respectively as sections of $\Lambda^2T^*M \otimes \mathrm{End}(TM) \otimes T^*M$ and $T^*M \otimes \mathrm{End}(TM) \otimes T^*M$. In both cases, the actions . are again those induced by the natural actions of $\Gamma(\mathrm{End}(TM))$ on tensors.

We summarize the above observations in a proposition:

Proposition 3.2.2 (Tractor tensor integrability conditions, odd dimension $n \geq 3$)**.** *Let $ABC_1 \cdots C_s$ be a multi-index specialized according to the splitting $Q \leftrightarrow (0, q, \infty)$ induced by a choice of representative $g \in c$, and let $\widetilde{g}$ be an ambient metric of c. If n odd, then for any parallel tractor tensor χ,*

$$[[(\widetilde{\nabla}^s\widetilde{R})|_{\mathcal{G}}].\chi]_{ABC_1\cdots C_S\mathbf{I}} = 0,$$

defining a (necessary) condition for a tractor tensor to be parallel, and the resulting condition is natural.

3.2.2 Even dimension

Now suppose $n > 2$ even. The Parallel Tractor Extension Theorem (Theorem 2.1.2) guarantees for a parallel tractor tensor χ the existence of an ambient extension $\widetilde{\chi}$ of χ satisfying $\widetilde{\nabla}\widetilde{\chi} = O(\rho^{n/2-1})$. We may proceed as in the odd case, with two caveats: First, since $\widetilde{\nabla}\widetilde{\chi}$ may only vanish to finite degree, the left-hand side of the tractor equation $[(\widetilde{\nabla}^s\widetilde{R})|_{\mathcal{G}}].\chi = 0$ can only be specialized in ways that do not involve taking too many derivatives transverse to $\mathcal{G}$: Commuting covariant derivatives gives

$$\begin{cases} (\widetilde{R}.\widetilde{\chi})_{AB\mathbf{I}} = O(\rho^{n/2-1}), & A \neq \infty, B \neq \infty \\ (\widetilde{R}.\widetilde{\chi})_{\infty B\mathbf{I}} = O(\rho^{n/2-2}), & B \neq \infty \end{cases}.$$

We are thus only guaranteed that $[(\nabla^s \widetilde{R})|_{\mathcal{G}}.\chi]_{AB\mathbf{I}C_1\cdots C_s} = 0$ if ∞ occurs in the multi-index $ABC_1 \cdots C_s$ at most $\frac{n}{2} - 2$ times, so in this discussion we henceforth assume that inequality holds. Second, for a general even-dimensional conformal structure, some components of the restrictions of covariant derivatives of the ambient curvature depend on the choice of ambient metric. Restricting to $M \subset \widetilde{M}$ and expanding gives

$$\begin{aligned} (\widetilde{R}.\widetilde{\chi})_{AB\mathbf{I},C_1\cdots C_s} &= \left(-\sum_{u=1}^{r} \widetilde{R}_{AB}{}^{Q}{}_{I_u} \widetilde{\chi}_{I_1\cdots I_{u-1}QI_{u+1}\cdots I_r} \right)_{,C_1\cdots C_s} \\ &= -\sum_{u=1}^{r} \widetilde{R}_{AB}{}^{Q}{}_{I_u,C_1\cdots C_s} \widetilde{\chi}_{I_1\cdots I_{u-1}QI_{u+1}\cdots I_r}. \end{aligned}$$

Letting L be the index that replaces Q when it is lowered; then, the maximum strength of an index of a component $\widetilde{R}_{ABLI_u,C_1\cdots C_s}$ independent of the ambiguity is $3 + ||ABC_1 \cdots C_s||$, and [FG, Proposition 6.2] shows that this component is independent of the choice of ambient metric if $||ABLI_u, C_1 \cdots C_s|| \leq n+1$. These observations allow us to generalize Proposition 3.2.2 to arbitrary dimension $n > 2$.

Proposition 3.2.3 (Tractor tensor integrability conditions, general dimension $n \geq 3$)**.** *Let χ be a parallel tractor tensor, let $ABC_1 \cdots C_s$ be a multi-index specialized according to the splitting $Q \leftrightarrow (0, q, \infty)$ induced by a choice of representative $g \in c$, and let $\widetilde{g}$ be an ambient metric of c. If n odd, or if $n > 2$ even and $||ABC_1 \cdots C_s|| \leq n - 2$, then*

$$[[(\widetilde{\nabla}^s \widetilde{R})|_{\mathcal{G}}].\chi]_{ABC_1\cdots C_S\mathbf{I}} = 0.$$

Moreover, the expression on the left-hand side is independent of the choice of ambient metric.

Proof. By the above observations, the conclusion holds if we additionally assume that ∞ can occur in $ABC_1 \cdots C_S$ at most $\frac{n}{2} - 2$ times. By the condition on the strength in the hypothesis, it can occur at most $\frac{n}{2} - 1$ times. If it holds exactly $\frac{n}{2} - 1$ times, then the remaining indices must all be zero, in which case the component is zero by Proposition 1.2.13. $\square$

For example, if $n = 4$, then this construction does not yield any conditions for multi-indices containing ∞. By the discussion before the proposition, the above formula for the conditions for a tractor form associated to the multi-index $ABC = abc$ do not apply for $n = 4$ (by the proposition, however, they do apply for $n = 3$ and $n \geq 5$); indeed, the tensor Y that occurs in those formulas is not defined for $n = 4$.

3.3 Outlook: Parallel extension and special holonomy

The Parallel Tractor Theorem and its application to G_2 holonomy raise a battery of new questions. For example, Berger's list contains the exceptional holonomy group $\mathrm{Spin}(3, 4)$, whose role in 8-dimensional geometry is formally analogous to that of G_2 in 7-dimensional geometry. This similarity is especially clear in terms of the constructions in Chapter 1 of this dissertation. Example 1.1.25 shows that generic 3-plane fields $\mathbb{E}$ on 6-manifolds N (here, $\mathbb{E}$ is generic if it satisfies $[\mathbb{E}, \mathbb{E}] = TN$) can be realized as (normal, regular) parabolic geometries of type $(\mathfrak{spin}(3, 4), Q)$. Here, $\mathrm{SO}(3, 4)$ acts naturally and transitively on the 6-dimensional projective subvariety $\mathrm{Gr}_0(3, \mathbb{R}^{3,4})$ of the hyperbolic Grassmannian $\mathrm{Gr}(3, \mathbb{R}^{3,4})$ comprising the null 3-planes in $\mathbb{R}^{3,4}$, and Q is the stabilizer of an arbitrary plane under that action. The flat model of this geometry is a $\mathrm{SO}(3, 4)$-invariant (nondegenerate) 3-plane field on $\mathrm{Gr}_0(3, \mathbb{R}^{3,4})$.

Bryant employed the natural inclusion $\mathrm{Spin}(3, 4) \hookrightarrow \mathrm{SO}(4, 4)$ to assign to any generic 3-plane field $\mathbb{E}$ on a 6-manifold N a canonical signature-$(3, 3)$ conformal structure $c_{\mathbb{E}}$ on N [Bry06]. One can then show that, analogously to Hammerl and Sagerschnig's construction (Theorem 3.1.1), if N is oriented, then the tractor bundle of $c_{\mathbb{E}}$ admits a nonzero parallel tractor 4-form of a particular algebraic type, namely, one whose stabilizer in $\mathrm{SO}(4, 4)$ is $\mathrm{Spin}(3, 4)$.

In view of this construction, it is natural to attempt to proceed as done in this dissertation for generic 2-plane fields on 5-manifolds: restrict to an arbitrary real-analytic plane field $\mathbb{E}$, construct a real-analytic ambient metric $\widetilde{g}_{\mathbb{E}}$ for the induced conformal structure $c_{\mathbb{E}}$, and apply the Parallel Tractor Extension Theorem to produce a parallel 4-form of the appropriate algebraic type on the ambient bundle, which would show that $\mathrm{Hol}(\widetilde{g}_{\mathbb{E}}) \leq \mathrm{Spin}(3, 4)$. (One surely expects again that generically, equality holds.) Because the underlying manifold

N has even dimension, however, both steps of this construction involve technical obstacles that do not occur in the analogous 2-plane field case. First, the Fefferman-Graham ambient construction only guarantees a priori the existence of an ambient metric of $c_{\mathbb{E}}$ Ricci-flat to order 3. For general conformal structures of dimension n, the existence of ambient metrics Ricci-flat beyond the critical order $\frac{1}{2}n$ is obstructed by the conformally-invariant natural obstruction tensor (see Subsection 1.2.5); in fact, this tensor is precisely the obstruction to the existence of ambient metrics Ricci-flat to infinite order in even dimension. So, a natural first step in analyzing the plane fields $\mathbb{E}$ in this setting is determining for which fields the obstruction vanishes. For such conformal structures, uniqueness of infinite-order ambient metrics still fails at order 3, though any such real-analytic ambient metric is completely determined up to diffeomorphism and extension by the choice of a single symmetric, tracefree 2-tensor at order 3 with prescribed divergence, but at least locally there always exists at least one choice; see [FG, Remark 3.12]).

Even for such a real-analytic ambient metric, the Parallel Tractor Extension Theorem (Theorem 2.1.2) a priori only guarantees the existence of an ambient extension of the tractor 4-form parallel to order 2. (Specializing Proposition 2.1.13 to the real-analytic case, however, shows the existence of a bona fide parallel extension is obstructed only at this order, that is, if there is an ambient extension of the tractor 4-form parallel to order 3, then there exists one parallel to infinite order). Then, as observed for general tractor tenors on even dimensional conformal manifolds in Section 2.2, the existence of a bona fide parallel ambient extension may depend on the choice of ambient metric. Thus, the extension problem for generic 3-plane manifolds on (oriented) 6-manifolds would involve analyzing the admissible prescriptions of the ambiguity in the ambient metrics of the conformal structures $c_{\mathbb{E}}$.

It is a priori plausible that some induced conformal structures $c_{\mathbb{E}}$ have non-vanishing obstruction tensors and hence do not admit ambient metrics Ricci-flat to order greater than 3. We can, however, partially address this issue by expanding attention to generalized ambient metrics $\widetilde{g}$, which are defined as ambient metrics except that (1) we only require that $\widetilde{g}$ need only be C^1 along $\mathcal{G}$ (but still insist that it be C^∞ elsewhere on the ambient manifold), and (2) $\widetilde{g}$ must be Ricci-flat to infinite order, in the sense that all components of derivatives of $\operatorname{Ric}(\widetilde{g})$ extend continuously across $\mathcal{G}$ and vanish there [FG]; an analogue of

Theorem 1.2.10 holds for such metrics. By [FG, Theorem 3.10], for any representative of a general conformal class of even dimension n, that conformal class admits a generalized ambient metric $\widetilde{g}$ in normal form with respect to g and which has an expansion of the form

$$\sum_{N=0}^{\infty} \widetilde{g}^{(N)} \cdot (\rho^{n/2} \log|\rho|)^N,$$

where the tensors $\widetilde{g}^{(N)}$ are smooth on the ambient space; if the underlying data is real-analytic, $\widetilde{g}$ is real-analytic in t, x, ρ, and $\rho^{n/2} \log|\rho|$ everywhere, and is real-analytic in t, x, and ρ everywhere except on $\mathcal{G}$. As for (standard) ambient metrics in even dimension, such a metric is determined to infinite order (and, when the analyticity hypothesis holds, up to extension and diffeomorphism preserving $\mathcal{G}$ pointwise) by the prescription of a single tensor subject to the same condition as before. Then, for any generalized ambient metric (satisfying the analyticity hypothesis) of a conformal class induced by a real-analytic plane field $\mathbb{E}$, one can look for parallel ambient extensions of the corresponding parallel tractor 4-form, the existence of which would yield metrics of holonomy contained in $\mathrm{Spin}(3,4)$. As in the obstruction-flat case, it is plausible that it admits extensions for some choices of such real-analytic ambient metrics but not others.

Several geometric structures that occur on even-dimensional pseudo-Riemannian manifolds are characterized by parallel tensors, including some whose existence is equivalent to containment of the holonomy of the metric in certain proper subgroups of $\mathrm{O}(p,q)$; indeed, with the exception of G_2, every irreducible holonomy group in Berger's List that can occur in indefinite signature occurs only in even dimension (recall that ambient metrics always have indefinite signature). One can then attempt to construct metrics with these additional structures using conformal structures and parallel extension as described for metrics of holonomy contained in $\mathrm{Spin}(3,4)$ as above.

For example, Čap and Gover have shown that a conformal structure has conformal holonomy contained in $\mathrm{SU}(\frac{p}{2}, \frac{q}{2})$—or equivalently, its tractor bundle admits a parallel complex structure $\mathbb{J}$ compatible with the tractor metric—iff it is (locally) equivalent to the Fefferman conformal structure of a CR manifold of hypersurface type [ČG10, Section 2.5]. So, one can then investigate the conditions under which such a $\mathbb{J}$, regarded as an adjoint tractor, on a real-analytic Fefferman conformal structure admits a parallel ambient extension to a suitably Ricci-flat (possibly generalized) ambient manifold, which would thus have holonomy contained in that special unitary group. Metrics with such holonomy are indefinite signature analogues of Calabi-Yau metrics. (Some Fefferman conformal structures are obstruction-flat and others are not, and so analysis of these structures in this setting varies as above.) See the upcoming paper [GW11] for more details of the relationship between Fefferman conformal structures and parallel tractor extension. In his Ph.D. thesis, Alt proved the analogous result for conformal holonomy contained in $\mathrm{Sp}(\frac{p}{4}, \frac{q}{4}) \leq \mathrm{SU}(\frac{p}{2}, \frac{q}{2})$ and Fefferman conformal structures of quaternionic contact structures [Alt08, Theorem 1]. Conformal holonomy of this type is characterized by the existence of three parallel complex structures on the tractor bundle that are compatible with the tractor metric and that satisfy the commutation relations of the imaginary quaternions i, j, and k. These relations are preserved by parallel extension, and so one may formulate the corresponding extension problem for producing ambient metrics with holonomy contained in $\mathrm{Sp}(\frac{p}{4}, \frac{q}{4})$; manifolds with such holonomy are called hyper-Kähler manifolds.

The cases discussed so far in fact already account for most of the groups on Berger's List that can occur as the holonomy of an ambient metric: If a Kähler manifold, that is, a manifold with holonomy contained in $\mathrm{U}(\frac{p}{2}, \frac{q}{2})$, is Ricci-flat, its canonical line bundle is flat, and thus its structure group can be reduced to a subgroup of $\mathrm{SL}(\frac{n}{2}, \mathbb{C})$, $n = p + q$. So, any Ricci-flat (possibly generalized) ambient manifold with holonomy contained in $\mathrm{U}(\frac{p}{2}, \frac{q}{2})$ in fact has holonomy contained in $\mathrm{U}(\frac{p}{2}, \frac{q}{2}) \cap \mathrm{SL}(\frac{n}{2}, \mathbb{C}) = \mathrm{SU}(\frac{p}{2}, \frac{q}{2})$ (also see [Nur08a]), and thus Čap and Gover's characterization applies. Likewise, if a quaternion-Kähler metric, that is, a metric with holonomy contained in $\mathrm{Sp}(\frac{p}{4}, \frac{q}{4}) \cdot \mathrm{Sp}(1)$, is Ricci-flat, its holonomy is actually contained in $\mathrm{Sp}(\frac{p}{4}, \frac{q}{4})$, and so Alt's analogous results apply to any suitably Ricci-flat (possibly generalized) ambient metric with such holonomy. So, the above considerations together account for all of the holonomy groups on Berger's list that can occur in indefinite signature except $\mathrm{SO}(n, \mathbb{C})$ and $\mathrm{G}_2^{\mathbb{C}}$.

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Appendix A

A.1 Nurowski's formula for a representative of the induced conformal class

Nurowski showed that for a function F for which $F_{qq} \neq 0$, the generic 2-plane field $\mathbb{D}_F = \operatorname{span}\{\partial_q, D\}$ induces a conformal class c_F containing the representative

$$
\begin{aligned}
g_F := {} & [(DF_{qq})^2 F_{qq}^2 + 6(DF_q)(DF_{qqq})F_{qq}^2 - 6(DF_{qqq})F_p F_{qq}^2 - 3(D^2F_{qq})F_{qq}^3 + 9(DF_{qp})F_{qq}^3 \\
& - 9F_{pp}F_{qq}^3 + 9(DF_{qz})F_q F_{qq}^3 - 18F_{pz}F_q F_{qq}^3 + 3(DF_z)F_{qq}^4 - 6(DF_q)F_{qq}^2 F_{qqp} \\
& + 6F_p F_{qq}^2 F_{qqp} - 8(DF_q)(DF_{qq})F_{qq}F_{qqq} + 8(DF_{qq})F_p F_{qq} F_{qqq} \\
& + 3(D^2F_q)F_{qq}^2 F_{qqq} - 3(DF_p)F_{qq}^2 F_{qqq} - 3(DF_z)F_q F_{qq}^2 F_{qqq} + 4(DF_q)^2 F_{qqq}^2 \\
& - 8(DF_q)F_p F_{qqq}^2 - 3(DF_q)^2 F_{qq} F_{qqqq} + 4F_p^2 F_{qqq}^2 + 6(DF_q)F_p F_{qq} F_{qqqq} \\
& - 3F_p^2 F_{qq} F_{qqqq} - 6(DF_q)F_q F_{qq}^2 F_{qqz} + 6F_p F_q F_{qq}^2 F_{qqz} - 3(DF_q)F_{qq}^3 F_{qz} \\
& + 12F_p F_{qq}^3 F_{qz} + 3F_{qq}^2 F_{qqq} F_y - 6(DF_{qqq})F_q F_{qq}^2 F_z + 4(DF_{qq})F_{qq}^3 F_z \\
& + 6F_q F_{qq}^2 F_{qqp} F_z + 8(DF_{qq})F_q F_{qq} F_{qqq} F_z - 4(DF_q)F_{qq}^2 F_{qqq} F_z - 9F_{qp}F_{qq}^3 F_z \\
& + F_p F_{qq}^2 F_{qqq} F_z - 8(DF_q)F_q F_{qqq}^2 F_z + 8F_p F_q F_{qqq}^2 F_z + 6(DF_q)F_q F_{qq} F_{qqqq} F_z \\
& - 6F_p F_q F_{qq} F_{qqqq} F_z + 18F_{qq}^3 F_{qy} + 6F_q^2 F_{qq}^2 F_{qqz} F_z + 3F_q F_{qq}^3 F_{qz} F_z - 2F_{qq}^4 F_z^2 \\
& + F_q F_{qq}^2 F_{qqq} F_z^2 + 4F_q^2 F_{qqq}^2 F_z^2 - 3F_q^2 F_{qq} F_{qqqq} F_z^2 - 9F_q^2 F_{qq}^3 F_{zz}](\widetilde{\omega}^1)^2 \\
& + [6(DF_{qqq})F_{qq}^2 - 6F_{qq}^2 F_{qqp} - 8(DF_{qq})F_{qq}F_{qqq} + 8(DF_q)F_{qqq}^2 - 8F_p F_{qqq}^2 \\
& - 6(DF_q)F_{qq}F_{qqqq} + 6F_p F_{qq} F_{qqqq} - 6F_q F_{qq}^2 F_{qqz} + 6F_{qq}^3 F_{qz} + 2F_{qq}^2 F_{qqq} F_z \\
& - 8F_q F_{qqq}^2 F_z + 6F_q F_{qq} F_{qqqq} F_z]\widetilde{\omega}^1 \widetilde{\omega}^2 + [10(DF_{qq})F_{qq}^3 - 10(DF_q)F_{qq}^2 F_{qqq} \\
& + 10F_p F_{qq}^2 F_{qqq} - 10F_{qq}^4 F_z + 10F_q F_{qq}^2 F_{qqq} F_z]\widetilde{\omega}^1 \widetilde{\omega}^3 + 30F_{qq}^4 \widetilde{\omega}^1 \widetilde{\omega}^4 \\
& + [30(DF_q)F_{qq}^3 - 30F_p F_{qq}^3 - 30F_q F_{qq}^3 F_z]\widetilde{\omega}^1 \widetilde{\omega}^5 + [4F_{qqq}^2 - 3F_{qq}F_{qqqq}](\widetilde{\omega}^2)^2 \\
& - 10F_{qq}^2 F_{qqq} \widetilde{\omega}^2 \widetilde{\omega}^3 + 30F_{qq}^3 \widetilde{\omega}^2 \widetilde{\omega}^5 - 20F_{qq}^4 (\widetilde{\omega}^3)^2, \qquad \text{(A.1)}
\end{aligned}
$$

where the coframe $(\widetilde{\omega}^a)$ is defined by

$$\begin{aligned}
\widetilde{\omega}^1 &:= dy - p\,dx \\
\widetilde{\omega}^2 &:= (dz - F\,dx) - F_q(dp - q\,dx) \\
\widetilde{\omega}^3 &:= dp - q\,dx \\
\widetilde{\omega}^4 &:= dq \\
\widetilde{\omega}^5 &:= dx.
\end{aligned}$$

This coframe is adapted to $\mathbb{D}_F$ in the sense that $(\bar{\theta}^a)$ defined in Subsection 1.5.1 is, namely that $\ker\{\widetilde{\omega}^1, \widetilde{\omega}^2, \widetilde{\omega}^3\} = \mathbb{D}$ and $\ker\{\widetilde{\omega}^1, \widetilde{\omega}^2\} = \mathbb{D}^\perp$. As in Subsection 1.5.2, D denotes

$$D := \partial_x + p\partial_y + q\partial_p + F\partial_z.$$

A.2 Tensorial data for Leistner and Nurowski's examples

We give here some tensorial data collected by Leistner and Nurowski for the class of examples studied in [LN10] and recorded here in Example 1.6.13; this data was reported in the appendix of [LN10] and is used here in the proof of Theorem 3.1.7.

For arbitrary parameters $\mathbf{a} = (a_0, \ldots, a_6) \in \mathbb{R}^7$ and $b \in \mathbb{R}$, define $F[\mathbf{a}, b] \in C^\infty(\mathbb{R}^5)$ by

$$F[\mathbf{a}, b](x, y, p, q, z) = q^2 + a_0 + a_1 p + a_2 p^2 + a_3 p^3 + a_4 p^4 + a_5 p^5 + a_6 p^6 + bz.$$

Any 2-plane field $\mathbb{D}_{F[\mathbf{a},b]} \subset T\mathbb{R}^5$ is generic, and so it defines a conformal class $c_{F[\mathbf{a},b]}$ (see Subsection 1.5.3) containing the representative $g_{F[\mathbf{a},b]}$ (A.1).

Define

$$\begin{aligned}
A_1 &:= \tfrac{1}{2^{1/3}}(a_1 + 2a_2 p + 3a_3 p^2 + 4a_4 p^3 + 5a_5 p^4 + 6a_6 p^5 + 2bq) \\
A_2 &:= \tfrac{1}{45\cdot 2^{2/3}}(9a_2 + 27a_3 p + 54a_4 p^2 + 90a_5 p^3 + 135a_6 p^4 + 2b^2) \\
A_3 &:= \tfrac{9}{20\cdot 2^{2/3}}(a_3 + 4a_4 p + 10a_5 p^2 + 20a_6 p^3) \\
A_4 &:= \tfrac{9}{10}(a_4 + 5a_5 p + 15a_6 p^2) \\
A_5 &:= \tfrac{27}{4\cdot 2^{1/3}}(a_5 + 6a_6 p).
\end{aligned}$$

Then, the Weyl tensor W_{abcd} of the representative $e^{-\frac{4b}{3}x}g_{F[\mathbf{a},b]} \in c_{F[\mathbf{a},b]}$ is given in a suitable coframe $(\widehat{\theta}^a)$ by

$$\begin{aligned}
W_{1214} &= -A_4 \\
W_{1313} &= -2A_4 \\
W_{1314} &= \tfrac{2^{4/3}}{3^{1/2}}(3 \cdot 2^{1/3}A_4 q - A_3 b) \\
W_{1414} &= \tfrac{1}{3}(27A_2^2 - 12 \cdot 2^{1/3}A_1 A_3 - 6 \cdot 2^{2/3}A_2 b^2 + 40A_3 bq - 24 \cdot 2^{1/3}A_4 q^2) \\
W_{1415} &= A_3 \\
W_{1424} &= A_3 \\
W_{1524} &= A_3,
\end{aligned}$$

and all components not determined by these by symmetry are zero. The Cotton tensor C_{abc} of $e^{-\frac{4b}{3}x}g_{F[\mathbf{a},b]}$ is given in the coframe $(\widehat{\theta}^a)$ by

$$\begin{aligned}
C_{113} &= -\tfrac{1}{3^{1/2}}A_5 \\
C_{114} &= \tfrac{2^{1/3}}{3}(A_4 b + 2^{4/3}A_5 q) \\
C_{314} &= -\tfrac{1}{3^{1/2}}A_4 \\
C_{413} &= -\tfrac{1}{3^{1/2}}A_4 \\
C_{414} &= \tfrac{2^{2/3}}{3}A_4 q,
\end{aligned}$$

where again all components not determined by these by symmetry are zero.

VITA

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